Electricity and Magnetics II

Lecture No.(12)- Semester 2

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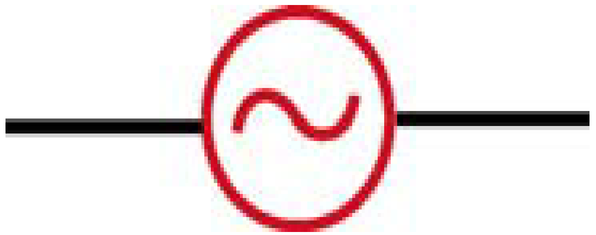
Alternating-Current Circuits دوائر التيار المتناوب

12.1 AC Sources

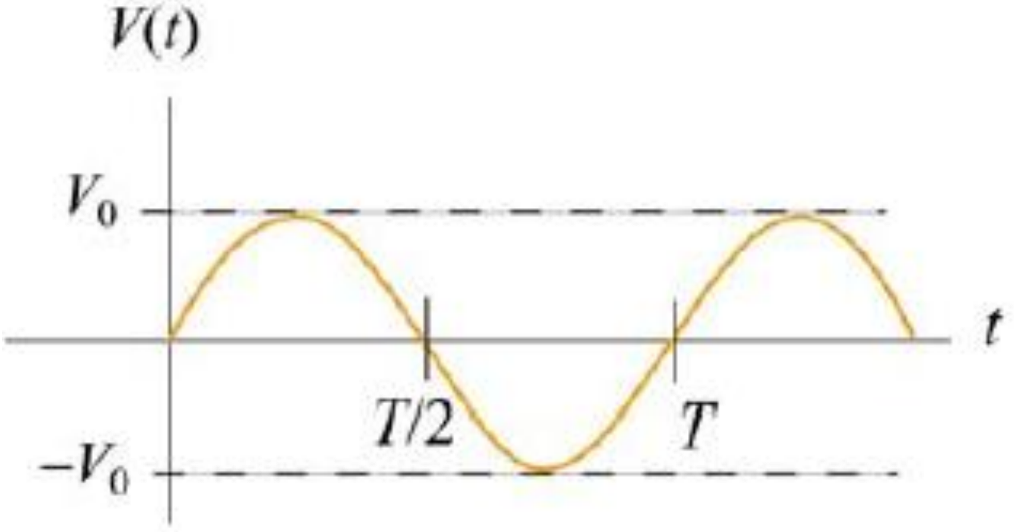
we learned that changing magnetic flux can induce an emf according to Faraday's law of induction. In particular, if a coil rotates in the presence of a magnetic field, the induced emf varies sinusoidally with time and leads to an alternating current (AC), and provides a source of AC power. The symbol for an AC voltage source is



An example of an AC source is

$$V(t) = V_0 \sin \omega t \qquad (12.1.1)$$

where the maximum value V_0 is called the *amplitude*. The voltage varies between V_0 and $-V_0$ since a sine function varies between +1 and −1. A graph of voltage as a function of time is shown in Figure



The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t' = t + T$ where T is the *period*. The *frequency*, f, defined as $f = 1/T$, has the unit of inverse seconds (s^{-1}), or hertz (Hz). The angular frequency is defined to be $\omega = 2\pi f$.

When a voltage source is connected to an *RLC* circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called driven or forced oscillations.

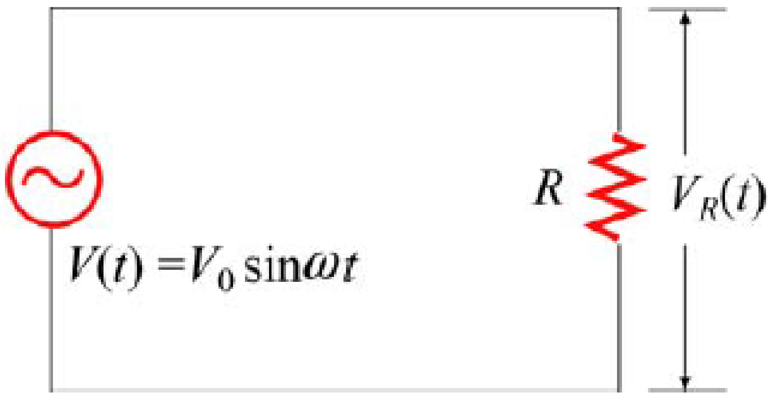
After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current, written as

$$I(t) = I_0 \sin(\omega t - \phi) \tag{12.1.2}$$

will oscillate with the same frequency as the voltage source, with an amplitude I_0 and phase ϕ that depends on the driving frequency.

12.2.1 Purely Resistive load

Consider a purely resistive circuit with a resistor connected to an AC generator, as shown in Figure 12.2.1. (As we shall see, a purely resistive circuit corresponds to infinite capacitance $C = \infty$ and zero inductance $L = 0$.)



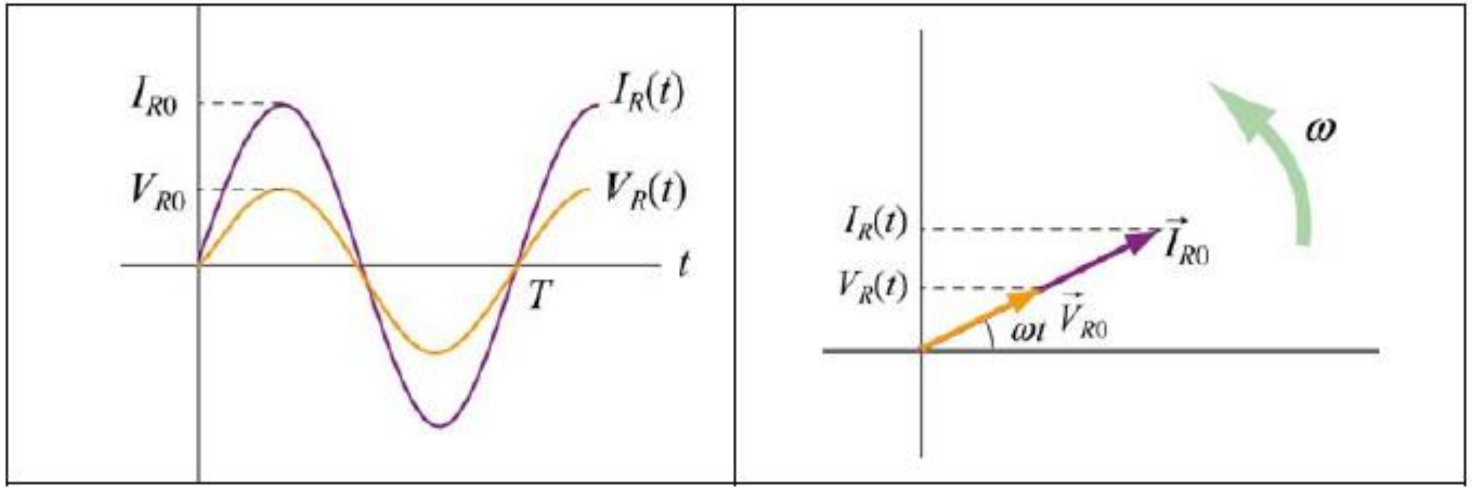
Applying Kirchhoff's loop rule yields

$$V(t) - V_R(t) = V(t) - I_R(t)R = 0 \tag{12.2.1}$$

where $V_R(t) = I_R(t)R$ is the instantaneous voltage drop across the resistor. The instantaneous current in the resistor is given by

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_{R0}\sin\omega t}{R} = I_{R0}\sin\omega t \tag{12.2.2}$$

where $V_{R0} = V_0$, and $I_{R0} = V_{R0}/R$ is the maximum current. Comparing Eq. (12.2.2) with Eq. (12.1.2), we find $\phi = 0$, which means that $I_R(t)$ and $V_R(t)$ are in phase with each other, meaning that they reach their maximum or minimum values at the same time. The time dependence of the current and the voltage across the resistor is depicted in Figure 12.2.2(a).



(a) Time dependence of $I_R(t)$ and $V_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $I_R(t)$ and $V_R(t)$ can also be represented with a phasor diagram, as shown in Figure 12.2.2(b). A phasor is a rotating vector having the following properties:

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.

(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

We shall denote a phasor with an arrow above it. The phasor $\vec{V}_{R0}$ has a constant magnitude of V_{R0}. Its projection along the vertical direction is $V_{R0}\sin\omega t$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\vec{I}_{R0}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period can be obtained as:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\ dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T}\ dt = 0 \qquad (12.2.3)$$

This average vanishes because

$$\langle \sin\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\ dt = 0 \qquad (12.2.4)$$

Similarly, one may find the following relations useful when averaging over one period:

$$\begin{aligned}
\langle \cos \omega t \rangle &= \frac{1}{T}\int_0^T \cos \omega t \ dt = 0 \\
\langle \sin \omega t \cos \omega t \rangle &= \frac{1}{T}\int_0^T \sin \omega t \cos \omega t \ dt = 0 \\
\langle \sin^2 \omega t \rangle &= \frac{1}{T}\int_0^T \sin^2 \omega t \ dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} \\
\langle \cos^2 \omega t \rangle &= \frac{1}{T}\int_0^T \cos^2 \omega t \ dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}
\end{aligned} \tag{12.2.5}$$

From the above, we see that the average of the square of the current is non-vanishing:

$$\langle I_R^2(t) \rangle = \frac{1}{T}\int_0^T I_R^2(t) dt = \frac{1}{T}\int_0^T I_{R0}^2 \sin^2 \omega t \ dt = I_{R0}^2 \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} I_{R0}^2 \tag{12.2.6}$$

It is convenient to define the root-mean-square (rms) current as

$$I_{\mathrm{rms}} = \sqrt{\langle I_R^2(t) \rangle} = \frac{I_{R0}}{\sqrt{2}} \tag{12.2.7}$$

In a similar manner, the rms voltage can be defined as

$$V_{\mathrm{rms}} = \sqrt{\langle V_R^2(t) \rangle} = \frac{V_{R0}}{\sqrt{2}} \tag{12.2.8}$$

The rms voltage supplied to the domestic wall outlets in the United States is $V_{\mathrm{rms}} = 120\ \mathrm{V}$ at a frequency $f = 60\ \mathrm{Hz}$.

The power dissipated in the resistor is

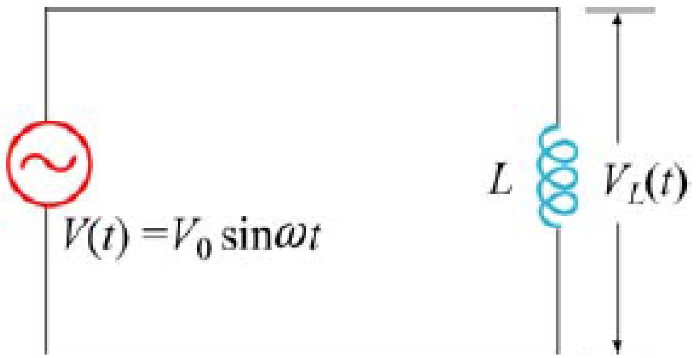
$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R \tag{12.2.9}$$

from which the average over one period is obtained as:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{R0}^2 R = I_{\text{rms}}^2 R = I_{\text{rms}}V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} \tag{12.2.10}$$

Purely Inductive Load

As we shall see below, a purely inductive circuit corresponds to infinite capacitance $C=\infty$ and zero resistance $R=0$. Applying the modified Kirchhoff's rule for inductors, the circuit equation reads



$$V(t)-V_L(t)=V(t)-L\frac{dI_L}{dt}=0 \qquad (12.2.11)$$

which implies

$$\frac{dI_L}{dt}=\frac{V(t)}{L}=\frac{V_{L0}}{L}\sin\omega t \qquad (12.2.12)$$

where $V_{L0}=V_0$. Integrating over the above equation, we find

$$I_L(t)=\int dI_L=\frac{V_{L0}}{L}\int\sin\omega t\ dt=-\left(\frac{V_{L0}}{\omega L}\right)\cos\omega t=\left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t-\frac{\pi}{2}\right) \qquad (12.2.13)$$

where we have used the trigonometric identity

$$-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right) \tag{12.2.14}$$

for rewriting the last expression. Comparing Eq. (12.2.14) with Eq. (12.1.2), we see that the amplitude of the current through the inductor is

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L} \tag{12.2.15}$$

where

$$X_L = \omega L \tag{12.2.16}$$

is called the *inductive reactance*. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

By comparing Eq. (12.2.14) to Eq. (12.1.2), we also find the phase constant to be

$$\phi = +\frac{\pi}{2} \tag{12.2.17}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.4 below.

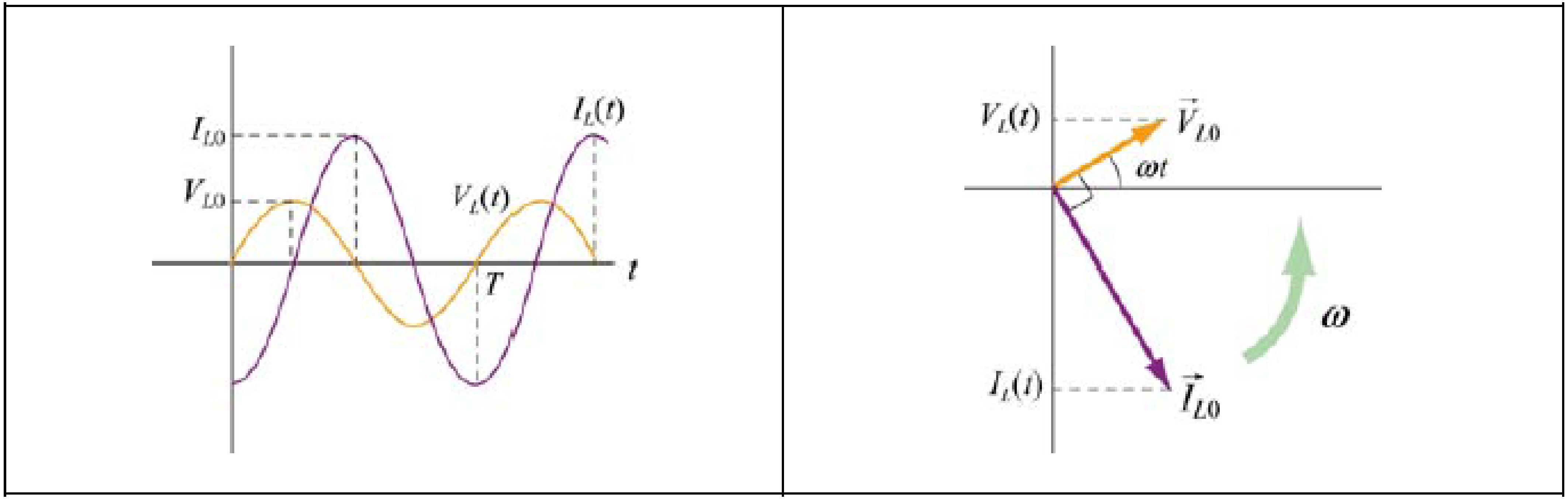


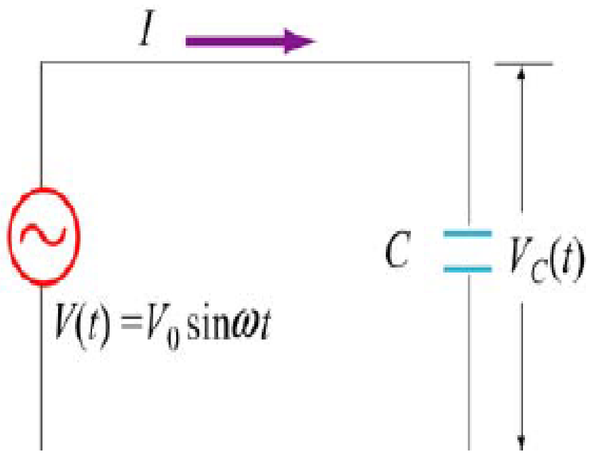
Figure 12.2.4 (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\phi = \pi / 2$; it reaches its maximum value after $V_L(t)$ does by one quarter of a cycle. Thus, we say that

The current lags voltage by $\pi / 2$ in a purely inductive circuit

Purely Capacitive Load

In the purely capacitive case, both resistance R and inductance L are zero. The circuit diagram is shown in Figure 12.2.5.



Again, Kirchhoff's voltage rule implies

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0 \tag{12.2.18}$$

which yields

$$Q(t) = CV(t) = CV_C(t) = CV_{C0} \sin \omega t \tag{12.2.19}$$

where $V_{C0} = V_0$. On the other hand, the current is

$$I_C(t) = +\frac{dQ}{dt} = \omega C V_{C0} \cos \omega t = \omega C V_{C0} \sin\left(\omega t + \frac{\pi}{2}\right) \tag{12.2.20}$$

where we have used the trigonometric identity

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right) \tag{12.2.21}$$

The above equation indicates that the maximum value of the current is

$$I_{C0} = \omega C V_{C0} = \frac{V_{C0}}{X_C} \tag{12.2.22}$$

where

$$X_C = \frac{1}{\omega C} \tag{12.2.23}$$

is called the *capacitance reactance*. It also has SI units of ohms and represents the effective resistance for a purely capacitive circuit. Note that X_C is inversely proportional to both C and ω, and diverges as ω approaches zero.

By comparing Eq. (12.2.21) to Eq. (12.1.2), the phase constant is given by

$$\phi = -\frac{\pi}{2} \tag{12.2.24}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.6 below.

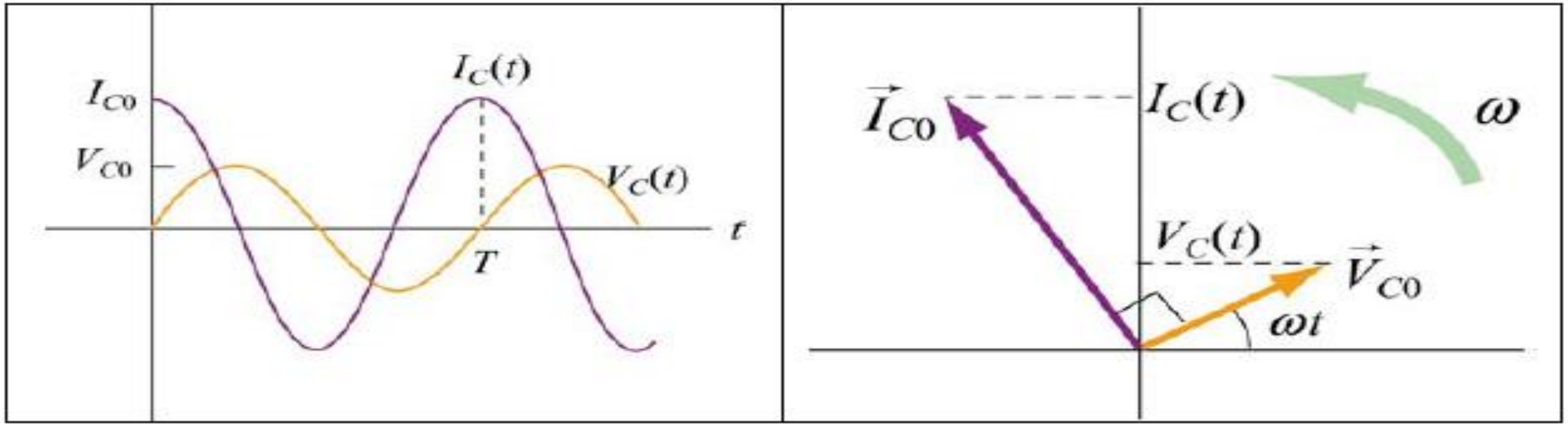


Figure 12.2.6 (a) Time dependence of $I_C(t)$ and $V_C(t)$ across the capacitor. (b) Phasor diagram for the capacitive circuit.

Notice that at $t = 0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\phi = \pi / 2$). Thus, we say that

The current leads the voltage by $\pi/2$ in a capacitive circuit

12.3 The *RLC* Series Circuit

Consider now the driven series *RLC* circuit shown in Figure 12.3.1.

Applying Kirchhoff's loop rule, we obtain

$$V(t)-V_R(t)-V_L(t)-V_C(t)=V(t)-IR-L\frac{dI}{dt}-\frac{Q}{C}=0 \qquad (12.3.1)$$

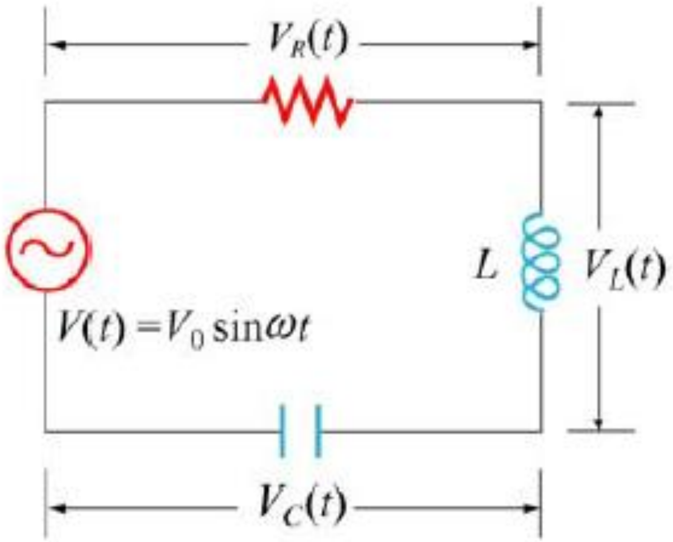


Figure 12.3.1 Driven series *RLC* Circuit

which leads to the following differential equation:

$$L\frac{dI}{dt}+IR+\frac{Q}{C}=V_0\sin\omega t \tag{12.3.2}$$

Assuming that the capacitor is initially uncharged so that $I=+dQ/dt$ is proportional to the *increase* of charge in the capacitor, the above equation can be rewritten as

$$\boxed{L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V_0\sin\omega t} \tag{12.3.3}$$

One possible solution to Eq. (12.3.3) is

$$Q(t)=Q_0\cos(\omega t-\phi) \tag{12.3.4}$$

where the amplitude and the phase are, respectively,

$$Q_0 = \frac{V_0 / L}{\sqrt{(R\omega / L)^2 + (\omega^2 - 1/LC)^2}} = \frac{V_0}{\omega\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$= \frac{V_0}{\omega\sqrt{R^2 + (X_L - X_C)^2}} \qquad (12.3.5)$$

and

$$\tan\phi = \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right) = \frac{X_L - X_C}{R} \qquad (12.3.6)$$

The corresponding current is

$$I(t)=+\frac{dQ}{dt}=I_0\sin(\omega t-\phi) \quad (12.3.7)$$

with an amplitude

$$I_0=-Q_0\omega=-\frac{V_0}{\sqrt{R^2+(X_L-X_C)^2}} \quad (12.3.8)$$

Notice that the current has the same amplitude and phase at all points in the series *RLC* circuit. On the other hand, the instantaneous voltage across each of the three circuit elements *R*, *L* and *C* has a different amplitude and phase relationship with the current, as can be seen from the phasor diagrams shown in Figure 12.3.2.

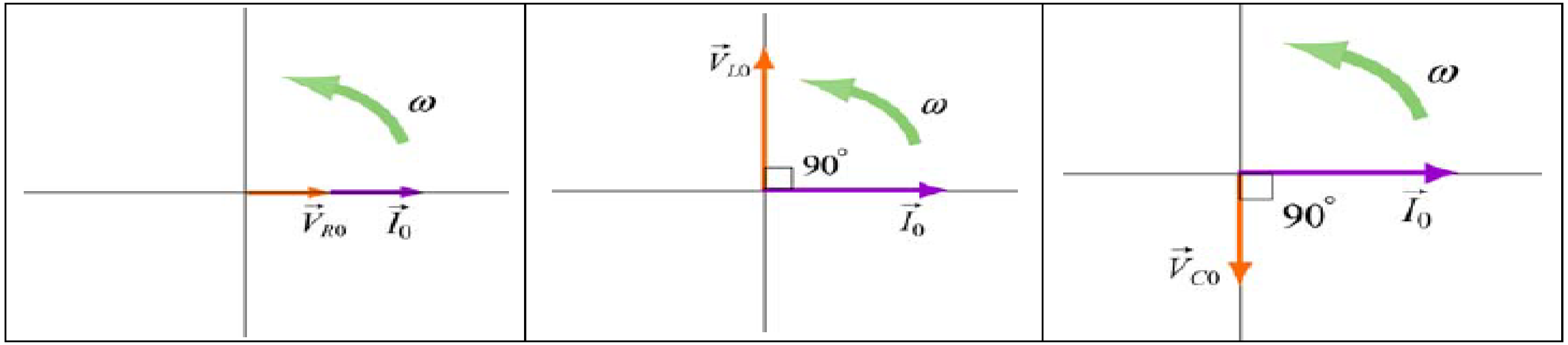


Figure 12.3.2 Phasor diagrams for the relationships between current and voltage in (a) the resistor, (b) the inductor, and (c) the capacitor, of a series *RLC* circuit.

From Figure 12.3.2, the instantaneous voltages can be obtained as:

$$\begin{aligned} V_R(t) &= I_0 R \sin \omega t = V_{R0} \sin \omega t \\ V_L(t) &= I_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_{L0} \cos \omega t \\ V_C(t) &= I_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -V_{C0} \cos \omega t \end{aligned} \tag{12.3.9}$$

where

$$V_{R0} = I_0 R, \quad V_{L0} = I_0 X_L, \quad V_{C0} = I_0 X_C \tag{12.3.10}$$

are the amplitudes of the voltages across the circuit elements. The sum of all three voltages is equal to the instantaneous voltage supplied by the AC source:

$$V(t) = V_R(t) + V_L(t) + V_C(t) \tag{12.3.11}$$

Using the phasor representation, the above expression can also be written as

$$\vec{V}_0 = \vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0} \tag{12.3.12}$$

as shown in Figure 12.3.3 (a). Again we see that current phasor $\vec{I}_0$ leads the capacitive voltage phasor $\vec{V}_{C0}$ by $\pi/2$ but lags the inductive voltage phasor $\vec{V}_{L0}$ by $\pi/2$. The three voltage phasors rotate counterclockwise as time passes, with their relative positions fixed.

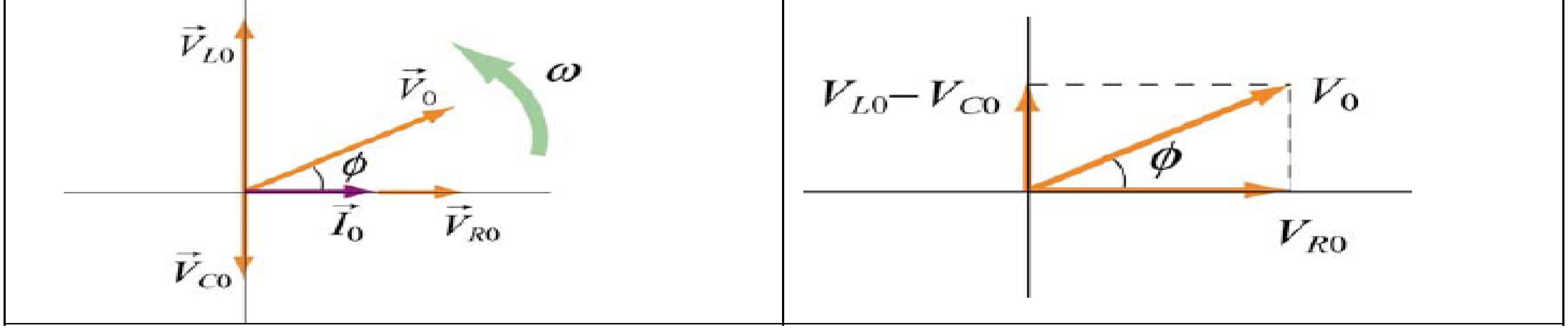


Figure 12.3.3 (a) Phasor diagram for the series *RLC* circuit. (b) voltage relationship

The relationship between different voltage amplitudes is depicted in Figure 12.3.3(b). From the Figure, we see that

$$\begin{aligned} V_0 &= |\vec{V}_0| = |\vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0}| = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} \\ &= \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2} \\ &= I_0 \sqrt{R^2 + (X_L - X_C)^2} \end{aligned} \qquad (12.3.13)$$

which leads to the same expression for I_0 as that obtained in Eq. (12.3.7).

It is crucial to note that the maximum amplitude of the AC voltage source V_0 is not equal to the sum of the maximum voltage amplitudes across the three circuit elements:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0} \qquad (12.3.14)$$

This is due to the fact that the voltages are not in phase with one another, and they reach their maxima at different times.