The Induced Current in a Motor

Q1/ A motor contains a coil with a total resistance of 10 V and is supplied by a voltage of 120 V. When the motor is running at its maximum speed, the back emf is 70 V.

(A) Find the current in the coil at the instant the motor is turned on.

(B) Find the current in the coil when the motor has reached maximum speed.

$$I = \frac{\mathcal{E}}{R} = \frac{120\ \text{V}}{10\ \Omega} = 12\ \text{A}$$

$$I = \frac{\mathcal{E} - \mathcal{E}_{\text{back}}}{R} = \frac{120\ \text{V} - 70\ \text{V}}{10\ \Omega} = \frac{50\ \text{V}}{10\ \Omega} = 5.0\ \text{A}$$

Self-Inductance and the Modified Kirchhoff's Loop Rule

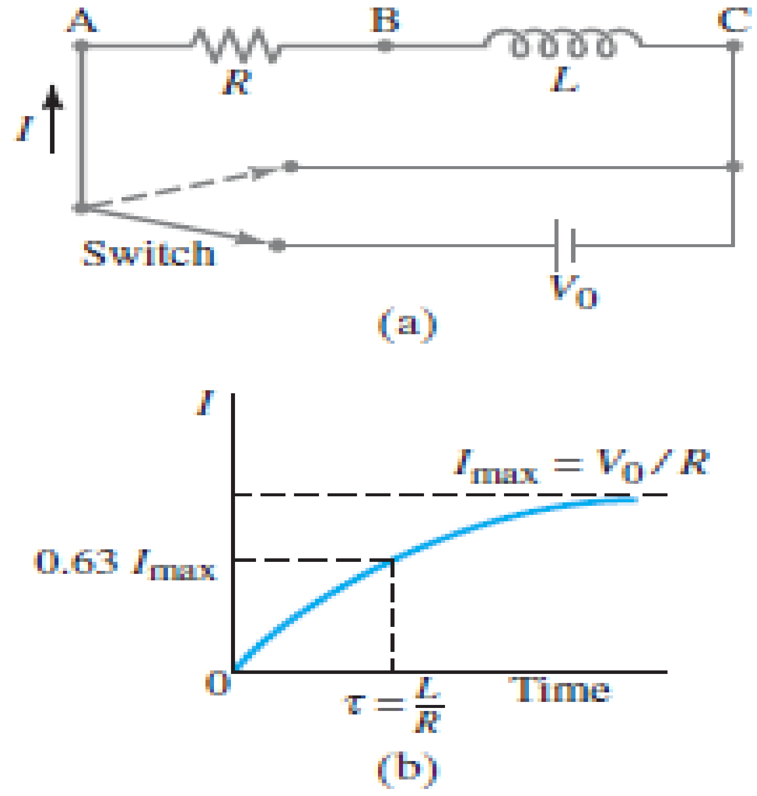
The addition of time-changing magnetic fields to simple circuits means that the closed line integral of the electric field around a circuit is no longer zero. Instead, we have, for any open surface

Any circuit where the current changes with time will have time-changing magnetic fields, and therefore induced electric fields we introduce time-changing magnetic fields, the electric potential difference between two points in our circuit is no longer well-defined, because when the line integral of the electric field around a closed loop is nonzero, the potential difference between two points, say a and b, is no longer independent of the path taken to get from a to b. That is, the electric field is no longer a conservative field, and the electric potential is no longer an appropriate concept, since we can no longer write E

the negative gradient of a scalar potential. However, we can still write down in a E straightforward fashion the equation that determines the behavior of a circuit

LR Circuits

Any inductor will have some resistance. We represent this situation by drawing its inductance L and its resistance R separately, as in Fig. 6a. The resistance R could also include any other resistance present in the circuit. Now we ask, what happens when a battery or other source of dc voltage V_0 is connected in series to such an LR circuit?



At the instant the switch connecting the battery is closed, the current starts to flow. It is opposed by the induced emf in the inductor which means point B in Fig. 6a is positive relative to point C. However, as soon as current starts to flow, there is also a voltage drop of magnitude IR across the resistance. Hence the voltage applied across the inductance is reduced and the current increases less rapidly. The current thus rises gradually as shown in Fig. 6b, and approaches the steady value $I_{max} = V_0/R_0$, for which all the voltage drop is across the resistance.

We can show this analytically by applying Kirchhoff's loop rule to the circuit of Fig. 6a. The emfs in the circuit are the battery voltage V_0 and the emf $\mathscr{E} = -L(dI/dt)$ in the inductor opposing the increasing current. Hence the sum of the potential changes around the loop is

$$V_0 - IR - L\frac{dI}{dt} = 0,$$

where I is the current in the circuit at any instant. We rearrange this to obtain

$$L \frac{dI}{dt} + RI = V_0.$$

This is a linear differential equation and can be integrated in the same way for an RC circuit. We rewrite Eq. 8 and then integrate:

$$\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{L}.$$

Then

$$-\frac{1}{R} \ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$$

or

$$I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) \qquad (9)$$

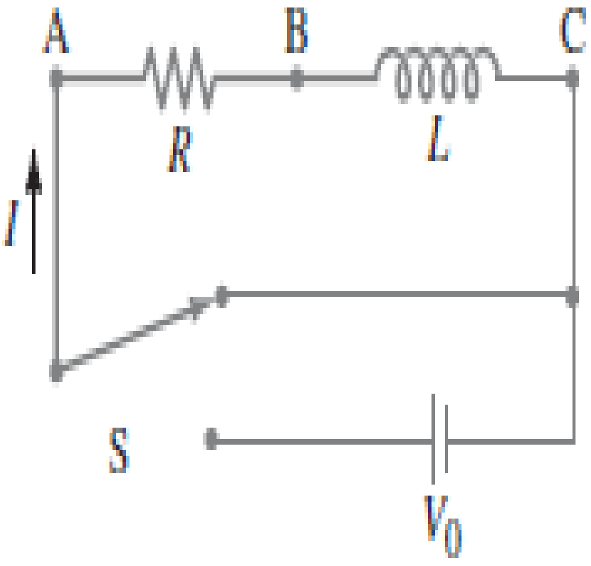
where

$$\tau = \frac{L}{R} \tag{10}$$

is the **time constant** of the *LR* circuit. The symbol τ represents the time required for the current I to reach $(1 - 1/e) = 0.63$ or 63% of its maximum value (V_0/R). Equation 9 is plotted in Fig. 6b. (Compare to the *RC* circuit.)

EXERCISE D Show that L/R does have dimensions of time.

Now let us flip the switch in Fig. 6a so that the battery is taken out of the circuit, and points A and C are connected together as shown in Fig. 7 at the moment when the switching occurs (call it $t = 0$) and the current is I_0. Then the differential equation (Eq. 8) becomes (since $V_0 = 0$):



$$L\frac{dI}{dt} + RI = 0.$$

We rearrange this equation and integrate:

$$\int_{I_0}^{I} \frac{dI}{I} = -\int_0^t \frac{R}{L}\,dt$$

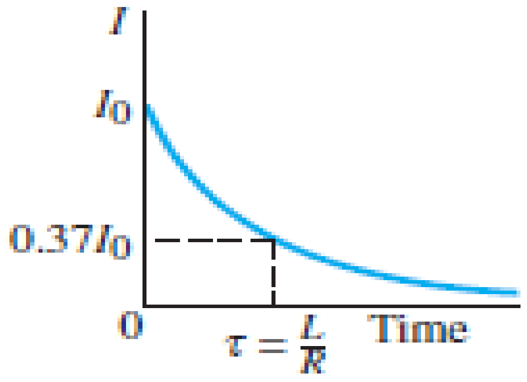
where $I = I_0$ at $t = 0$, and $I = I$ at time t.

We integrate this last equation to obtain

$$\ln\frac{I}{I_0} = -\frac{R}{L}t$$

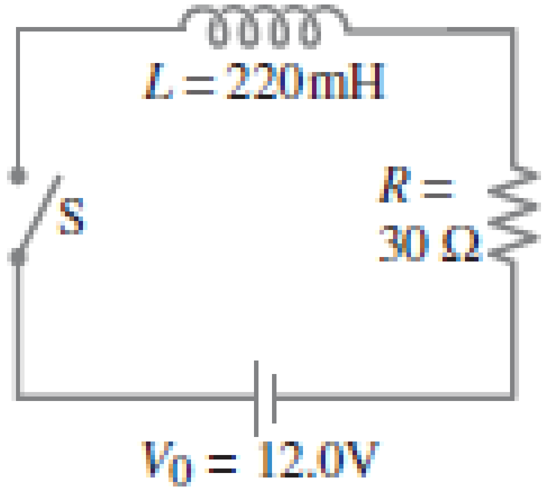
or

$$I = I_0 e^{-t/\tau}$$



where again the time constant is $\tau = L/R$. The current thus decays exponentially to zero

An *LR* circuit. At $t = 0$, a 12.0-V battery is connected in series with a 220-mH inductor and a total of 30-Ω resistance, as shown in Fig. 9. (*a*) What is the current at $t = 0$? (*b*) What is the time constant? (*c*) What is the maximum current? (*d*) How long will it take the current to reach half its maximum possible value? (*e*) At this instant, at what rate is energy being delivered by the battery, and (*f*) at what rate is energy being stored in the inductor's magnetic field?



SOLUTION (*a*) The current cannot instantaneously jump from zero to some other value when the switch is closed because the inductor opposes the change $(\mathscr{E}_L = -L(dI/dt))$. Hence just after the switch is closed, I is still zero at $t = 0$ and then begins to increase.

(*b*) The time constant is, from Eq. 10, $\tau = L/R = (0.22\text{ H})/(30\,\Omega) = 7.3\text{ ms}$.

(*c*) The current reaches its maximum steady value after a long time, when $dI/dt = 0$ so $I_{max} = V_0/R = 12.0\text{ V}/30\,\Omega = 0.40\text{ A}$.

(*d*) We set $I = \frac{1}{2}I_{max} = V_0/2R$ in Eq. 9, which gives us

$$1 - e^{-t/\tau} = \frac{1}{2}$$

or

$$e^{-t/\tau} = \frac{1}{2}.$$

We solve for t:

$$t = \tau \ln 2 = (7.3 \times 10^{-3}\text{ s})(0.69) = 5.0\text{ ms}.$$

(*e*) At this instant, $I = I_{max}/2 = 200\ \text{mA}$, so the power being delivered by the battery is

$$P = IV = (0.20\ \text{A})(12\ \text{V}) = 2.4\ \text{W}.$$

(*f*) From Eq. 6, the energy stored in an inductor L at any instant is

$$U = \tfrac{1}{2}LI^2$$

where I is the current in the inductor at that instant. The *rate* at which the energy changes is

$$\frac{dU}{dt} = LI\frac{dI}{dt}.$$

We can differentiate Eq. 9 to obtain dI/dt, or use the differential equation, Eq. 8, directly:

$$\frac{dU}{dt} = I\left(L\frac{dI}{dt}\right) = I(V_0 - RI)$$

$$= (0.20\ \text{A})[12\ \text{V} - (30\ \Omega)(0.20\ \text{A})] = 1.2\ \text{W}.$$

EXERCISE E A resistor in series with an inductor has a time constant of 10 ms. When the same resistor is placed in series with a 5-μF capacitor, the time constant is 5×10^{-6} s. What is the value of the inductor? (*a*) 5 μH; (*b*) 10 μH; (*c*) 5 mH; (*d*) 10 mH; (*e*) not enough information to determine it.

LC Circuits and Electromagnetic Oscillations

In any electric circuit, there can be three basic components: resistance, capacitance, and inductance, in addition to a source of emf. (There can also be more complex

components, such as diodes or transistors.) Capacitance C

tance C and an inductance, L, Fig. 10. This is an idealized circuit in which we assume there is no resistance; in the next Section we will include resistance. Let us suppose the capacitor in Fig. 10 is initially charged so that one plate has charge Q_0 and the other plate has charge $-Q_0$, and the potential difference across it is $V = Q/C$. Suppose that at $t = 0$, the switch is closed. The capacitor immediately begins to discharge. As it does so, the current I through the inductor increases. We now apply Kirchhoff's loop rule (sum of potential changes around a loop is zero):

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0.$$

Because charge leaves the positive plate on the capacitor to produce the current I as shown in Fig. 10, the charge Q on the (positive) plate of the capacitor is decreasing, so $I = -dQ/dt$. We can then rewrite the above equation as

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0. \tag{12}$$

This is a familiar differential equation. It has the same form as the equation for simple harmonic motion. The solution of Eq. 12 can be written as

$$Q = Q_0 \cos(\omega t + \phi) \tag{13}$$

where Q_0 and ϕ are constants that depend on the initial conditions. We insert Eq. 13 into Eq. 12, noting that $d^2Q/dt^2 = -\omega^2 Q_0 \cos(\omega t + \phi)$; thus

$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{1}{LC} Q_0 \cos(\omega t + \phi) = 0$$

or

$$\left(-\omega^2 + \frac{1}{LC}\right) \cos(\omega t + \phi) = 0.$$

This relation can be true for all times t only if $\left(-\omega^2 + 1/LC\right) = 0$, which tells us that

$$\omega = 2\pi f = \sqrt{\frac{1}{LC}}. \tag{14}$$

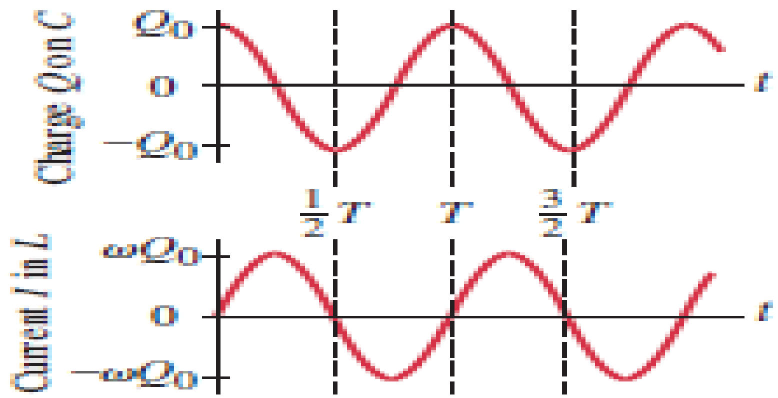
Equation 13 shows that the charge on the capacitor in an LC circuit oscillates sinusoidally. The current in the inductor is

$$I = -\frac{dQ}{dt} = \omega Q_0 \sin(\omega t + \phi)$$

$$= I_0 \sin(\omega t + \phi); \qquad (15)$$

so the current too is sinusoidal. The maximum value of I is $I_0 = \omega Q_0 = Q_0/\sqrt{LC}$. Equations 13 and 15 for Q and I when $\phi = 0$ are plotted in Fig. 11.

FIGURE 11 Charge Q and current I in an LC circuit. The period $T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$.



Now let us look at LC oscillations from the point of view of energy. The energy stored in the electric field of the capacitor at any time t is:

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\omega t + \phi).$$

The energy stored in the magnetic field of the inductor at the same instant is (Eq. 6)

$$U_B = \frac{1}{2}LI^2 = \frac{L\omega^2 Q_0^2}{2}\sin^2(\omega t + \phi) = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi)$$

where we used Eq. 14. If we let $\phi = 0$, then at times $t = 0$, $t = \frac{1}{2}T$, $t = T$, and so on (where T is the period $= 1/f = 2\pi/\omega$), we have $U_E = Q_0^2/2C$ and $U_B = 0$. That is, all the energy is stored in the electric field of the capacitor. But at $t = \frac{1}{4}T, \frac{3}{4}T$, and so on, $U_E = 0$ and $U_B = Q_0^2/2C$, and so all the energy is stored in the magnetic field of the inductor. At any time t, the total energy is

$$\begin{aligned} U = U_E + U_B &= \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 \\ &= \frac{Q_0^2}{2C}\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] = \frac{Q_0^2}{2C}. \end{aligned} \qquad (16)$$

EXAMPLE 7 ***LC* circuit.** A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at $t = 0$, to a 75-mH inductor. Determine: (*a*) the initial charge on the capacitor; (*b*) the maximum current; (*c*) the frequency f and period T of oscillation; and (*d*) the total energy oscillating in the system.

SOLUTION (*a*) The 500-V power supply, before being disconnected, charged the capacitor to a charge of

$$Q_0 = CV = (1.2 \times 10^{-9}\,\mathrm{F})(500\,\mathrm{V}) = 6.0 \times 10^{-7}\,\mathrm{C}.$$

(*b*) The maximum current, I_{max}, is (see Eqs. 14 and 15)

$$I_{max} = \omega Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{(6.0 \times 10^{-7}\,\mathrm{C})}{\sqrt{(0.075\,\mathrm{H})(1.2 \times 10^{-9}\,\mathrm{F})}} = 63\,\mathrm{mA}.$$

(*c*) Equation 14 gives us the frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{(2\pi\sqrt{LC})} = 17\,\mathrm{kHz},$$

and the period T is

$$T = \frac{1}{f} = 6.0 \times 10^{-5}\,\mathrm{s}.$$

(*d*) Finally the total energy (Eq. 16) is

$$U = \frac{Q_0^2}{2C} = \frac{(6.0 \times 10^{-7}\,\mathrm{C})^2}{2(1.2 \times 10^{-9}\,\mathrm{F})} = 1.5 \times 10^{-4}\,\mathrm{J}.$$

LC Oscillations with Resistance (LRC Circuit)

The LC circuit discussed in the previous Section is an idealization. There is always some resistance R in any circuit, and so we now discuss such a simple LRC circuit, Fig. 13.

Suppose again that the capacitor is initially given a charge Q_0 and the battery or other source is then removed from the circuit. The switch is closed at $t = 0$. Since there is now a resistance in the circuit, we expect some of the energy to be converted to thermal energy, and so we don't expect undamped oscillations as in a pure LC circuit. Indeed, if we use Kirchhoff's loop rule around this circuit, we obtain

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0,$$

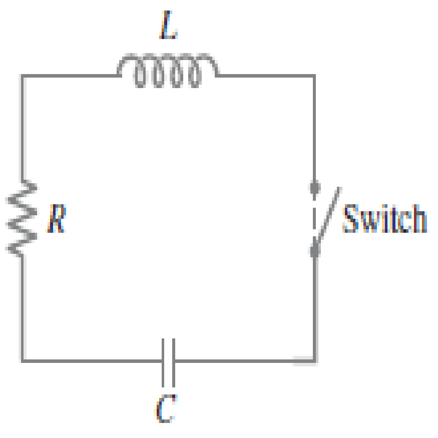


FIGURE 13 An LRC circuit.

which is the same equation we had in Section 5 with the addition of the voltage drop IR across the resistor. Since $I = -dQ/dt$, as we saw in Section 5, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0. \tag{17}$$

This second-order differential equation in the variable Q has precisely the same form as that for the damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

Hence we can analyze our LRC circuit in the same way as for damped harmonic motion. Our system may undergo damped oscillations, curve A in Fig. 14 (underdamped system), or it may be critically damped (curve B), or overdamped (curve C), depending on the relative values of R, L, and C. With m replaced by L, b by R, and k by C^{-1}, we find that the system will be underdamped when

$$R^2 < \frac{4L}{C},$$

and overdamped for $R^2 > 4L/C$. Critical damping (curve B in Fig. 14) occurs when $R^2 = 4L/C$. If R is smaller than $\sqrt{4L/C}$, the angular frequency, ω', will be

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \quad (18)$$

And the charge Q as a function of time will be

$$Q = Q_0 e^{-\frac{R}{2L}t} \cos(\omega' t + \phi) \quad (19)$$

where ϕ is a phase constant.