Electricity and Magnetics II

Lecture No.(3)- Semester 2

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2017-2018

Sources of Magnetic Fields

9.1 Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current I, the magnetic field at any point P due to the current can be calculated by adding up the magnetic field contributions, $d\vec{\mathbf{B}}$, from small segments of the wire $d\vec{\mathbf{s}}$, (Figure 9.1.1).

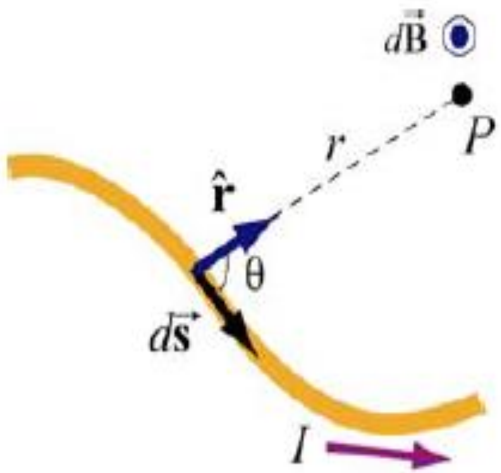

Figure 9.1.1 Magnetic field $d\vec{\mathbf{B}}$ at point P due to a current-carrying element $I\,d\vec{\mathbf{s}}$.

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $I\,d\vec{\mathbf{s}}$.

Let r denote as the distance form the current source to the field point P, and $\hat{\mathbf{r}}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{\mathbf{B}}$, from the current source, $I d\vec{\mathbf{s}}$,

$$\boxed{d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}} \qquad (9.1.1)$$

where μ_0 is a constant called the *permeability of free space*:

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A} \tag{9.1.2}$$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element dq:

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} \tag{9.1.3}$$

Adding up these contributions to find the magnetic field at the point P requires integrating over the current source,

$$\vec{\mathbf{B}} = \int_{\text{wire}} d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{9.1.4}$$

The integral is a vector integral, which means that the expression for $\vec{\mathbf{B}}$ is really three integrals, one for each component of $\vec{\mathbf{B}}$. The vector nature of this integral appears in the cross product $I\,d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Example 9.1: Magnetic Field due to a Finite Straight Wire

A thin, straight wire carrying a current I is placed along the x-axis, as shown in Figure 9.1.3. Evaluate the magnetic field at point P. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point P.

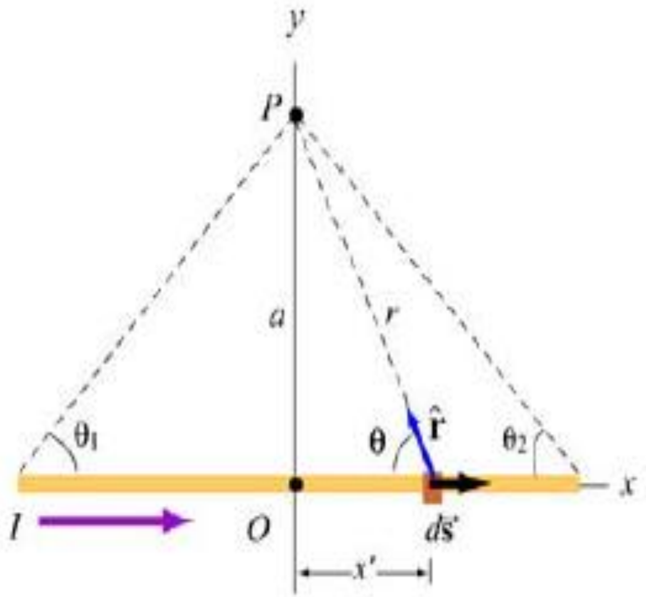

Figure 9.1.3 A thin straight wire carrying a current I.

Solution:

This is a typical example involving the use of the Biot-Savart law. We solve the problem using the methodology summarized in Section 9.10.

(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{\mathbf{s}} = dx'\hat{\mathbf{i}}$ carrying current I in the x-direction. The location of this source is represented by $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$.

(2) Field point (coordinates denoted with a subscript "P")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing P is $\vec{\mathbf{r}}_P = a\hat{\mathbf{j}}$.

(3) Relative position vector

The vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$ is a "relative" position vector which points from the source point to the field point. In this case, $\vec{\mathbf{r}} = a\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}$, and the magnitude $r = |\vec{\mathbf{r}}| = \sqrt{a^2 + x'^2}$ is the distance from between the source and P. The corresponding unit vector is given by

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{a\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{i}}$$

(4) The cross product $d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}$

The cross product is given by

$$d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}} = (dx'\,\hat{\mathbf{i}}) \times (-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (dx'\sin\theta)\,\hat{\mathbf{k}}$$

(5) Write down the contribution to the magnetic field due to $I d\vec{\mathbf{s}}$

The expression is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\,\sin\theta}{r^2}\hat{\mathbf{k}}$$

which shows that the magnetic field at P will point in the $+\hat{\mathbf{k}}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure 9.1.3, we have

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \quad \Rightarrow \quad dx = -a\csc^2\theta\, d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi}\frac{(-a\csc^2\theta\, d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we obtain

$$\boxed{B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2} \sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)} \tag{9.1.5}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add!

Let's examine the following cases:

(i) In the symmetric case where $\theta_2 = -\theta_1$, the field point P is located along the perpendicular bisector. If the length of the rod is $2L$, then $\cos\theta_1 = L/\sqrt{L^2+a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta_1 = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2+a^2}} \tag{9.1.6}$$

(ii) The infinite length limit $L \to \infty$

This limit is obtained by choosing $(\theta_1, \theta_2) = (0, 0)$. The magnetic field at a distance a away becomes

$$\boxed{B = \frac{\mu_0 I}{2\pi a}} \tag{9.1.7}$$

Example 9.2: Magnetic Field due to a Circular Current Loop

A circular loop of radius R in the xy plane carries a steady current I, as shown in Figure 9.1.6.

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z \hat{\mathbf{k}}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e., $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$

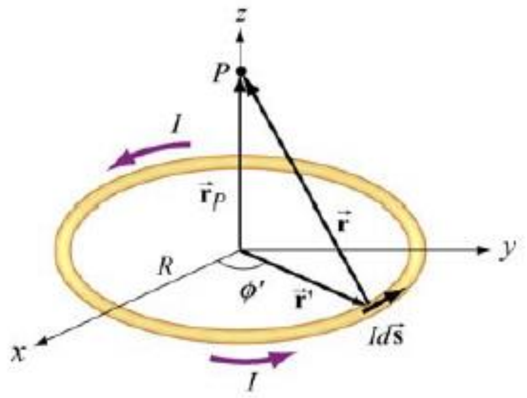

Figure 9.1.6 Magnetic field due to a circular loop carrying a steady current.

Solution:

(a) This is another example that involves the application of the Biot-Savart law. Again let's find the magnetic field by applying the same methodology used in Example 9.1.

(1) Source point

In Cartesian coordinates, the differential current element located at $\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ can be written as $Id\vec{\mathbf{s}} = I(d\vec{\mathbf{r}}'/d\phi')d\phi' = IRd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$.

(2) Field point

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$.

(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$

The relative position vector is given by

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,' = -R\cos\phi'\,\hat{\mathbf{i}} - R\sin\phi'\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{9.1.8}$$

and its magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + (-R\sin\phi')^2 + z^2} = \sqrt{R^2 + z^2} \tag{9.1.9}$$

is the distance between the differential current element and P. Thus, the corresponding unit vector from $Id\,\vec{\mathbf{s}}$ to P can be written as

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}')$ can be simplified as

$$\begin{aligned} d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}') &= R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}}\right)\times[-R\cos\phi'\,\hat{\mathbf{i}}-R\sin\phi'\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}] \end{aligned} \tag{9.1.10}$$

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$\begin{aligned} d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}=\frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}=\frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}')}{|\vec{\mathbf{r}}_P-\vec{\mathbf{r}}'|^3} \\ &= \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}\,d\phi' \end{aligned} \tag{9.1.11}$$

(6) Carrying out the integration

Using the result obtained above, the magnetic field at P is

$$\vec{\mathbf{B}} = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + R\,\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}\, d\phi' \tag{9.1.12}$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero:

$$B_x = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \int_0^{2\pi} \cos\phi'\, d\phi' = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \sin\phi' \Big|_0^{2\pi} = 0 \tag{9.1.13}$$

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \int_0^{2\pi} \sin\phi'\, d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \cos\phi' \Big|_0^{2\pi} = 0 \tag{9.1.14}$$

On the other hand, the z component is

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \qquad (9.1.15)$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments.

The behavior of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown in Figure 9.1.7:

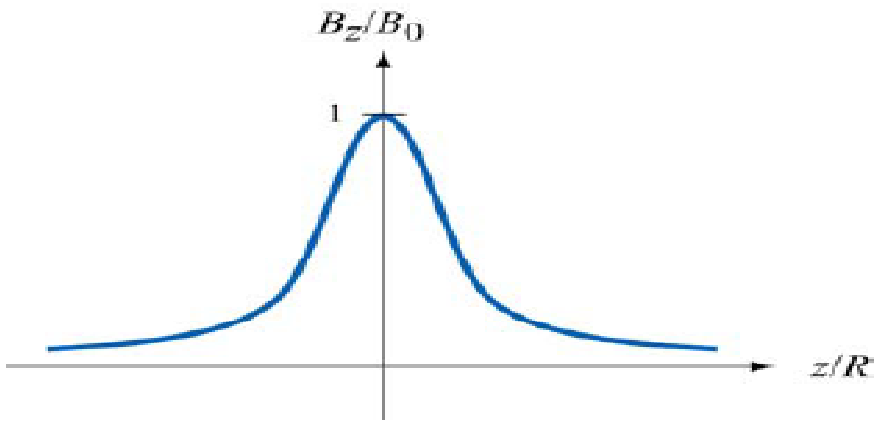

Figure 9.1.7 The ratio of the magnetic field, B_z / B_0, as a function of z / R

(b) If we place a magnetic dipole $\vec{\mu} = \mu_z \hat{\mathbf{k}}$ at the point P, as discussed in Chapter 8, due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\mu} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z}{dz} \right) \hat{\mathbf{k}} \tag{9.1.16}$$

Upon differentiating Eq. (9.1.15) and substituting into Eq. (9.1.16), we obtain

$$\vec{\mathbf{F}}_B = -\frac{3\mu_z \mu_0 I R^2 z}{2(R^2 + z^2)^{5/2}} \hat{\mathbf{k}} \tag{9.1.17}$$

Thus, the dipole is attracted toward the current-carrying ring. On the other hand, if the direction of the dipole is reversed, $\vec{\mu} = -\mu_z \hat{\mathbf{k}}$, the resulting force will be repulsive.