Electricity and Magnetics II

Lecture No.(1)- Semester 2

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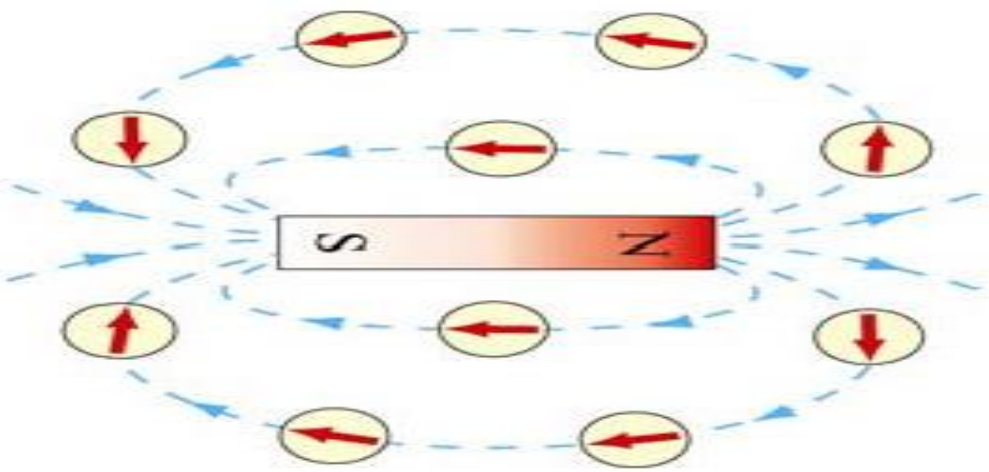
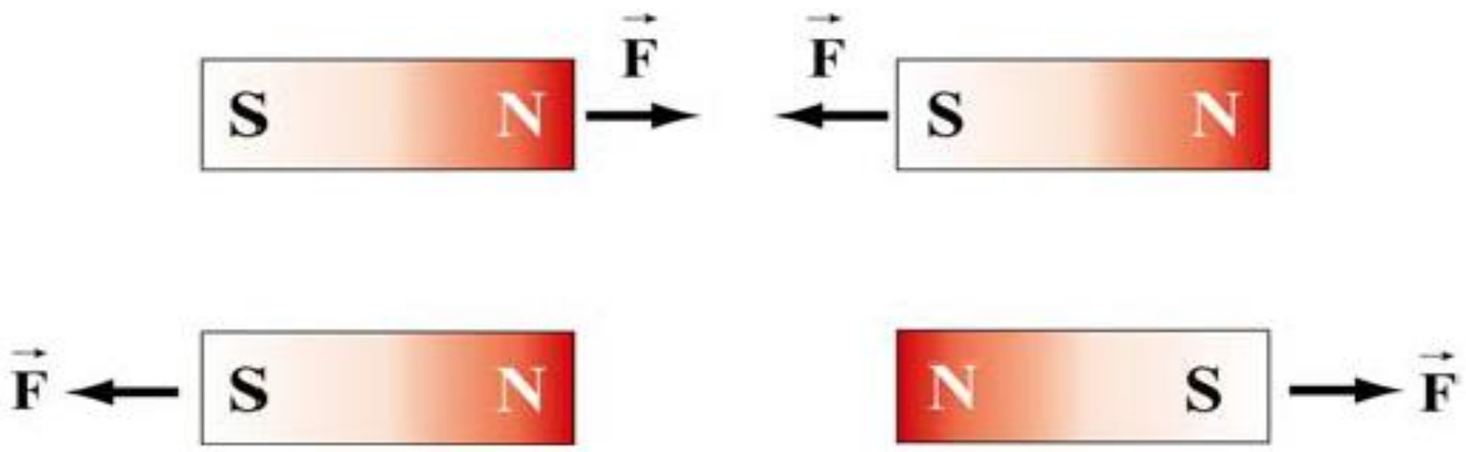
Introduction to Magnetic Fields

Introduction

We have seen that a charged object produces an electric field $\vec{E}$ at all points in space. In a similar manner, a bar magnet is a source of a magnetic field $\vec{B}$. This can be readily demonstrated by moving a compass near the magnet. The compass needle will line up along the direction of the magnetic field produced by the magnet, as depicted in Figure



Notice that the bar magnet consists of two poles, which are designated as the north (N) and the south (S). Magnetic fields are strongest at the poles. The magnetic field lines leave from the north pole and enter the south pole. When holding two bar magnets close to each other, the like poles will repel each other while the opposite poles attract (Figure



Unlike electric charges which can be isolated, the two magnetic poles always come in a pair. When you break the bar magnet, two new bar magnets are obtained, each with a north pole and a south pole (Figure 8.1.3). In other words, magnetic "monopoles" do not exist in isolation, although they are of theoretical interest.

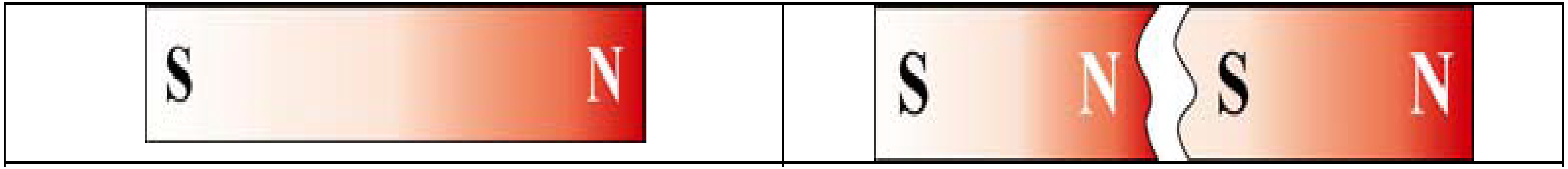


Figure 8.1.3 Magnetic monopoles do not exist in isolation

How do we define the magnetic field $\vec{\mathbf{B}}$? In the case of an electric field $\vec{\mathbf{E}}$, we have already seen that the field is defined as the force per unit charge:

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}_e}{q} \qquad (8.1.1)$$

However, due to the absence of magnetic monopoles, $\vec{\mathbf{B}}$ must be defined in a different way.

8.2 The Definition of a Magnetic Field

To define the magnetic field at a point, consider a particle of charge q and moving at a velocity $\vec{\mathbf{v}}$. Experimentally we have the following observations:

(1) The magnitude of the magnetic force $\vec{\mathbf{F}}_B$ exerted on the charged particle is proportional to both v and q.

(2) The magnitude and direction of $\vec{\mathbf{F}}_B$ depends on $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.

(3) The magnetic force $\vec{\mathbf{F}}_B$ vanishes when $\vec{\mathbf{v}}$ is parallel to $\vec{\mathbf{B}}$. However, when $\vec{\mathbf{v}}$ makes an angle θ with $\vec{\mathbf{B}}$, the direction of $\vec{\mathbf{F}}_B$ is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, and the magnitude of $\vec{\mathbf{F}}_B$ is proportional to $\sin\theta$.

(4) When the sign of the charge of the particle is switched from positive to negative (or vice versa), the direction of the magnetic force also reverses.

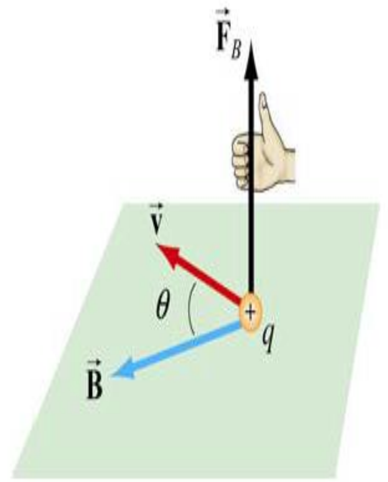


Figure 8.2.1 The direction of the magnetic force

The above observations can be summarized with the following equation:

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{8.2.1}$$

The above expression can be taken as the working definition of the magnetic field at a point in space. The magnitude of $\vec{\mathbf{F}}_B$ is given by

$$F_B = |q| vB \sin\theta \tag{8.2.2}$$

The SI unit of magnetic field is the tesla (T):

$$1 \ \text{tesla} = 1 \ \text{T} = 1 \ \frac{\text{Newton}}{(\text{Coulomb})(\text{meter/second})} = 1 \ \frac{\text{N}}{\text{C} \cdot \text{m/s}} = 1 \ \frac{\text{N}}{\text{A} \cdot \text{m}}$$

Another commonly used non-SI unit for $\vec{\mathbf{B}}$ is the *gauss* (G), where $1\text{T} = 10^4 \text{ G}$.

Note that $\vec{\mathbf{F}}_B$ is always perpendicular to $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, and cannot change the particle's speed v (and thus the kinetic energy). In other words, magnetic force cannot speed up or slow down a charged particle. Consequently, $\vec{\mathbf{F}}_B$ can do no work on the particle:

$$dW = \vec{\mathbf{F}}_B \cdot d\,\vec{\mathbf{s}} = q(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{v}}\,dt = q(\vec{\mathbf{v}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{B}}\,dt = 0 \tag{8.2.3}$$

The direction of $\vec{\mathbf{v}}$, however, can be altered by the magnetic force, as we shall see below.

8.3 Magnetic Force on a Current-Carrying Wire

We have just seen that a charged particle moving through a magnetic field experiences a magnetic force $\vec{\mathbf{F}}_B$. Since electric current consists of a collection of charged particles in motion, when placed in a magnetic field, a current-carrying wire will also experience a magnetic force.

Consider a long straight wire suspended in the region between the two magnetic poles. The magnetic field points out the page and is represented with dots (•). It can be readily demonstrated that when a downward current passes through, the wire is deflected to the left. However, when the current is upward, the deflection is rightward, as shown in Figure 8.3.1.

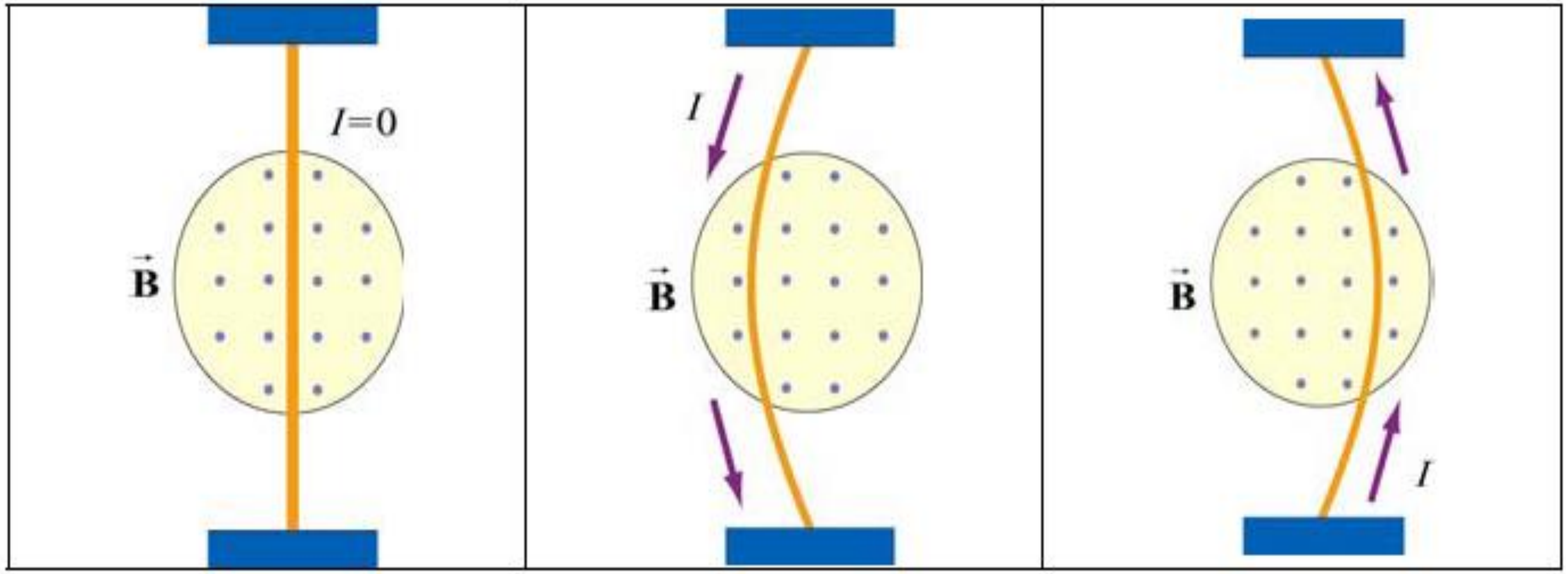


Figure 8.3.1 Deflection of current-carrying wire by magnetic force

To calculate the force exerted on the wire, consider a segment of wire of length ℓ and cross-sectional area A, as shown in Figure 8.3.2. The magnetic field points into the page, and is represented with crosses (X).

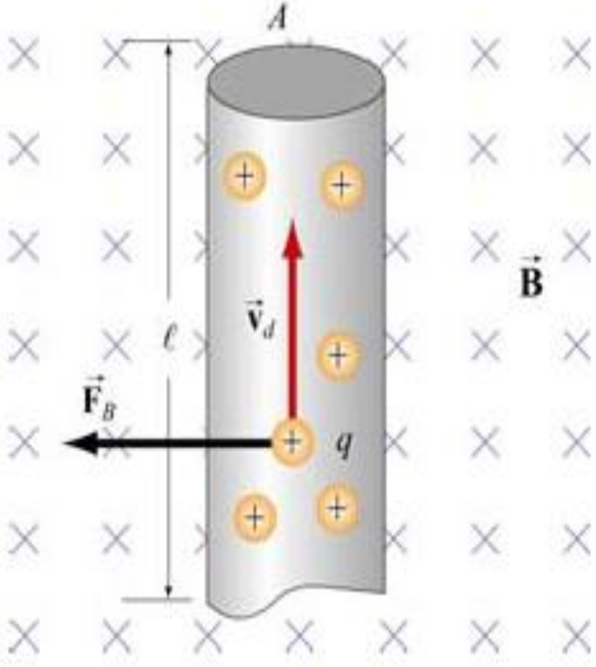


Figure 8.3.2 Magnetic force on a conducting wire

The charges move at an average drift velocity $\vec{\mathbf{v}}_d$. Since the total amount of charge in this segment is $Q_{\text{tot}} = q(nA\ell)$, where n is the number of charges per unit volume, the total magnetic force on the segment is

$$\vec{\mathbf{F}}_B = Q_{\text{tot}} \vec{\mathbf{v}}_d \times \vec{\mathbf{B}} = qnA\ell(\vec{\mathbf{v}}_d \times \vec{\mathbf{B}}) = I(\vec{\ell} \times \vec{\mathbf{B}}) \qquad (8.3.1)$$

where $I = nqv_d A$, and $\vec{\ell}$ is a *length vector* with a magnitude ℓ and directed along the direction of the electric current.

For a wire of arbitrary shape, the magnetic force can be obtained by summing over the forces acting on the small segments that make up the wire. Let the differential segment be denoted as $d\vec{\mathbf{s}}$ (Figure 8.3.3).

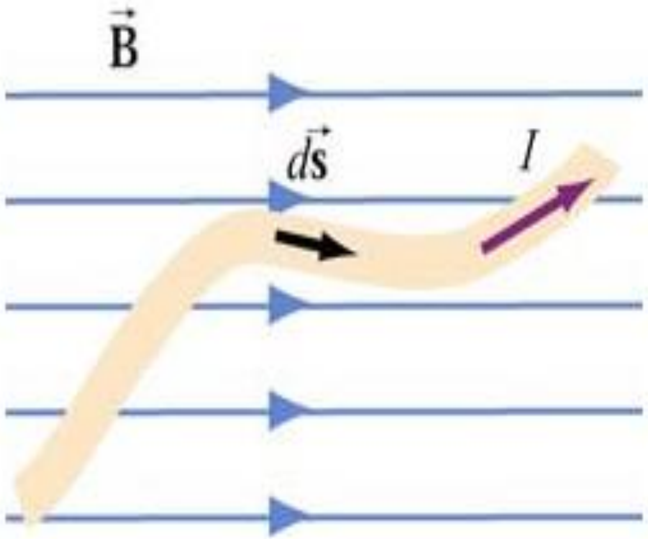


Figure 8.3.3 Current-carrying wire placed in a magnetic field

The magnetic force acting on the segment is

$$d\vec{\mathbf{F}}_B = I d\,\vec{\mathbf{s}} \times \vec{\mathbf{B}} \tag{8.3.2}$$

Thus, the total force is

$$\boxed{\vec{\mathbf{F}}_B = I \int_a^b d\,\vec{\mathbf{s}} \times \vec{\mathbf{B}}} \tag{8.3.3}$$

where a and b represent the endpoints of the wire.

As an example, consider a curved wire carrying a current I in a uniform magnetic field $\vec{\mathbf{B}}$, as shown in Figure 8.3.4.

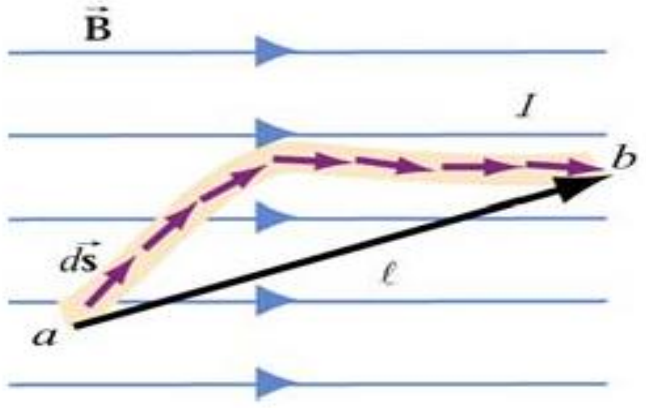


Figure 8.3.4 A curved wire carrying a current I.

Using Eq. (8.3.3), the magnetic force on the wire is given by

$$\vec{\mathbf{F}}_B = I\left(\int_a^b d\,\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} = I\vec{\ell}\times\vec{\mathbf{B}} \tag{8.3.4}$$

where $\vec{\ell}$ is the length vector directed from a to b. However, if the wire forms a closed loop of arbitrary shape (Figure 8.3.5), then the force on the loop becomes

$$\vec{\mathbf{F}}_B = I\left(\oint d\,\vec{\mathbf{s}}\right)\times\vec{\mathbf{B}} \tag{8.3.5}$$

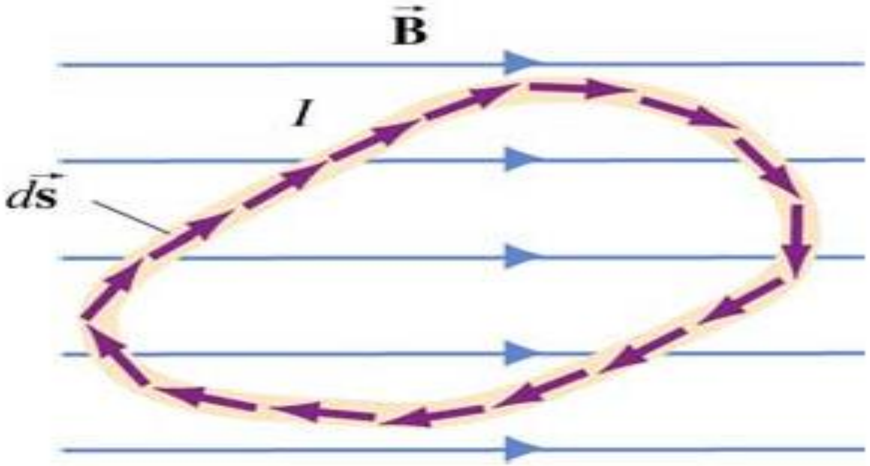


Figure 8.3.5 A closed loop carrying a current I in a uniform magnetic field.

Since the set of differential length elements $d\vec{\mathbf{s}}$ form a closed polygon, and their vector sum is zero, i.e., $\oint d\vec{\mathbf{s}} = 0$. The net magnetic force on a closed loop is $\vec{\mathbf{F}}_B = \vec{0}$.

Example 8.1: Magnetic Force on a Semi-Circular Loop

Consider a closed semi-circular loop lying in the *xy* plane carrying a current *I* in the counterclockwise direction, as shown in Figure 8.3.6.

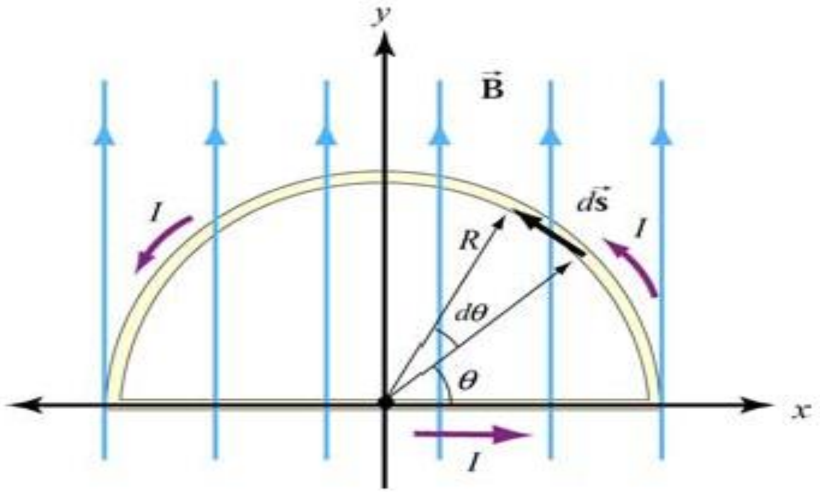


Figure 8.3.6 Semi-circular loop carrying a current *I*

A uniform magnetic field pointing in the $+y$ direction is applied. Find the magnetic force acting on the straight segment and the semicircular arc.

Solution:

Let $\vec{\mathbf{B}} = B\hat{\mathbf{j}}$ and $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ the forces acting on the straight segment and the semicircular parts, respectively. Using Eq. (8.3.3) and noting that the length of the straight segment is $2R$, the magnetic force is

$$\vec{\mathbf{F}}_1 = I(2R\,\hat{\mathbf{i}}) \times (B\,\hat{\mathbf{j}}) = 2IRB\,\hat{\mathbf{k}}$$

where $\hat{\mathbf{k}}$ is directed out of the page.

To evaluate $\vec{\mathbf{F}}_2$, we first note that the differential length element $d\,\vec{\mathbf{s}}$ on the semicircle can be written as $d\,\vec{\mathbf{s}} = ds\,\hat{\boldsymbol{\theta}} = Rd\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$. The force acting on the length element $d\,\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{F}}_2 = I d\,\vec{\mathbf{s}} \times \vec{\mathbf{B}} = IR\,d\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) \times (B\,\hat{\mathbf{j}}) = -IBR\sin\theta d\theta\,\hat{\mathbf{k}}$$

Here we see that $d\vec{\mathbf{F}}_2$ points into the page. Integrating over the entire semi-circular arc, we have

$$\vec{\mathbf{F}}_2 = -IBR\,\hat{\mathbf{k}} \int_0^{\pi} \sin\theta d\theta = -2IBR\,\hat{\mathbf{k}}$$

Thus, the net force acting on the semi-circular wire is

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \vec{\mathbf{0}}$$

This is consistent from our previous claim that the net magnetic force acting on a closed current-carrying loop must be zero.

8.4 Torque on a Current Loop

What happens when we place a rectangular loop carrying a current I in the xy plane and switch on a uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{i}}$ which runs parallel to the plane of the loop, as shown in Figure 8.4.1(a)?

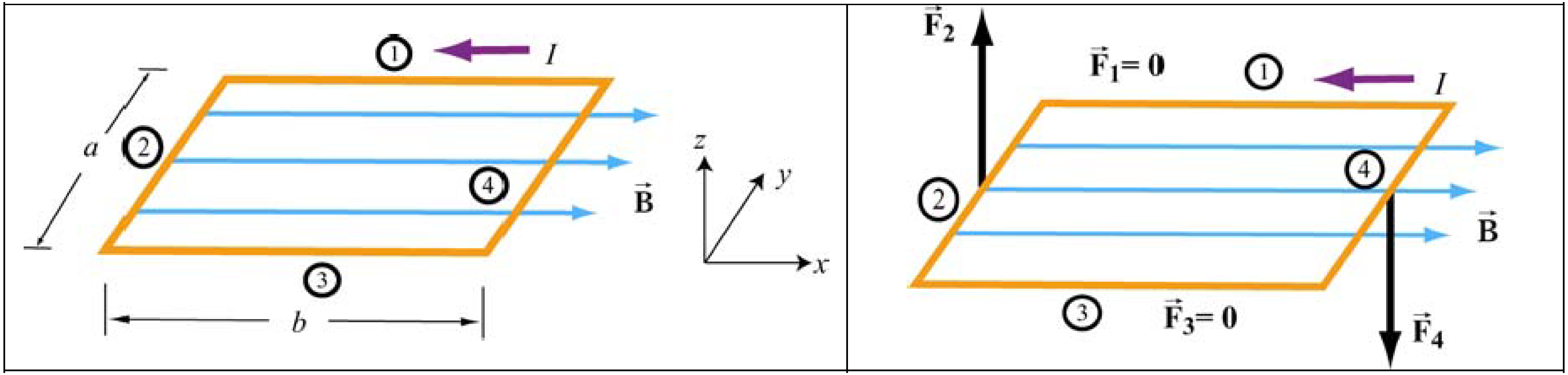


Figure 8.4.1 (a) A rectangular current loop placed in a uniform magnetic field. (b) The magnetic forces acting on sides 2 and 4.

From Eq. 8.4.1, we see the magnetic forces acting on sides 1 and 3 vanish because the length vectors $\vec{\ell}_1 = -b\hat{\mathbf{i}}$ and $\vec{\ell}_3 = b\hat{\mathbf{i}}$ are parallel and anti-parallel to $\vec{\mathbf{B}}$ and their cross products vanish. On the other hand, the magnetic forces acting on segments 2 and 4 are non-vanishing:

$$\begin{cases} \vec{\mathbf{F}}_2 = I(-a\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{i}}) = IaB\,\hat{\mathbf{k}} \\ \vec{\mathbf{F}}_4 = I(a\,\hat{\mathbf{j}})\times(B\,\hat{\mathbf{i}}) = -IaB\,\hat{\mathbf{k}} \end{cases} \tag{8.4.1}$$

with $\vec{\mathbf{F}}_2$ pointing out of the page and $\vec{\mathbf{F}}_4$ into the page. Thus, the net force on the rectangular loop is

$$\vec{\mathbf{F}}_{net} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}} \tag{8.4.2}$$

as expected. Even though the net force on the loop vanishes, the forces $\vec{\mathbf{F}}_2$ and $\vec{\mathbf{F}}_4$ will produce a torque which causes the loop to rotate about the *y*-axis (Figure 8.4.2). The torque with respect to the center of the loop is

$$\vec{\boldsymbol{\tau}} = \left(-\frac{b}{2}\hat{\mathbf{i}}\right)\times\vec{\mathbf{F}}_2 + \left(\frac{b}{2}\hat{\mathbf{i}}\right)\times\vec{\mathbf{F}}_4 = \left(-\frac{b}{2}\hat{\mathbf{i}}\right)\times\left(IaB\hat{\mathbf{k}}\right) + \left(\frac{b}{2}\hat{\mathbf{i}}\right)\times\left(-IaB\hat{\mathbf{k}}\right)$$
$$= \left(\frac{IabB}{2}+\frac{IabB}{2}\right)\hat{\mathbf{j}} = IabB\hat{\mathbf{j}} = IAB\hat{\mathbf{j}} \qquad (8.4.3)$$

where $A = ab$ represents the area of the loop and the positive sign indicates that the rotation is clockwise about the y-axis. It is convenient to introduce the area vector $\vec{\mathbf{A}} = A\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector in the direction normal to the plane of the loop. The direction of the positive sense of $\hat{\mathbf{n}}$ is set by the conventional right-hand rule. In our case, we have $\hat{\mathbf{n}} = +\hat{\mathbf{k}}$. The above expression for torque can then be rewritten as

$$\vec{\boldsymbol{\tau}} = I\vec{\mathbf{A}} \times \vec{\mathbf{B}} \tag{8.4.4}$$

Notice that the magnitude of the torque is at a maximum when $\vec{\mathbf{B}}$ is parallel to the plane of the loop (or perpendicular to $\vec{\mathbf{A}}$).

Consider now the more general situation where the loop (or the area vector $\vec{\mathbf{A}}$) makes an angle θ with respect to the magnetic field.

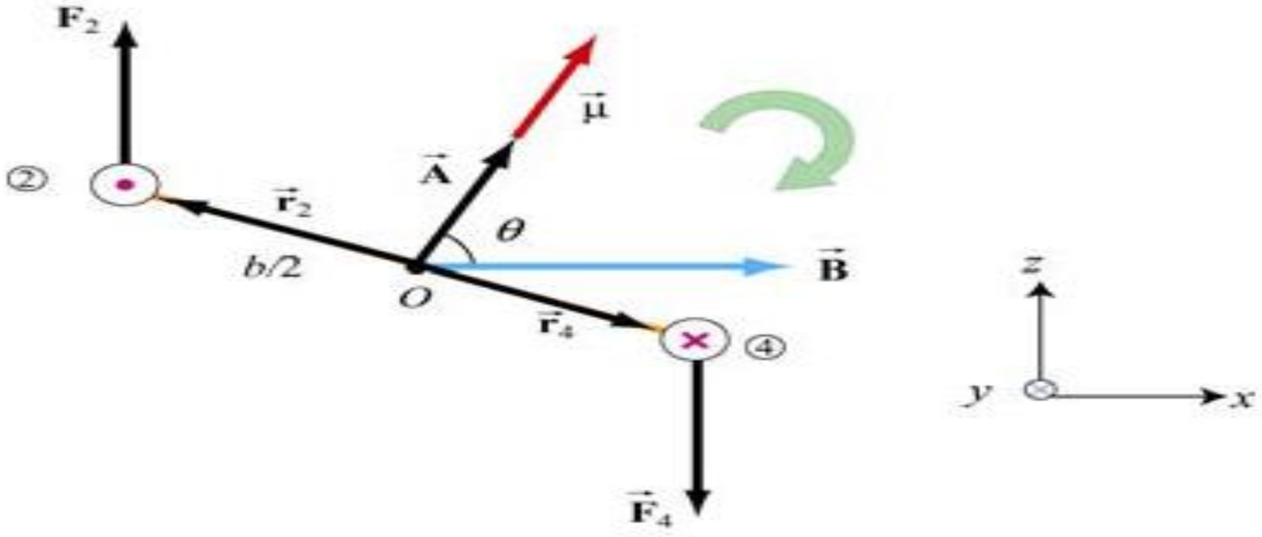


Figure 8.4.2 Rotation of a rectangular current loop

From Figure 8.4.2, the lever arms and can be expressed as:

$$\vec{\mathbf{r}}_2 = \frac{b}{2}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}\right) = -\vec{\mathbf{r}}_4 \qquad (8.4.5)$$

and the net torque becomes

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2 + \vec{\mathbf{r}}_4 \times \vec{\mathbf{F}}_4 = 2\vec{\mathbf{r}}_2 \times \vec{\mathbf{F}}_2 = 2\cdot\frac{b}{2}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}\right)\times\left(IaB\,\hat{\mathbf{k}}\right) \\ &= IabB\sin\theta\,\hat{\mathbf{j}} = I\vec{\mathbf{A}}\times\vec{\mathbf{B}}\end{aligned} \qquad (8.4.6)$$

For a loop consisting of N turns, the magnitude of the toque is

$$\tau = NIAB\sin\theta \tag{8.4.7}$$

The quantity $NI\vec{\mathbf{A}}$ is called the magnetic dipole moment $\vec{\boldsymbol{\mu}}$:

$$\boxed{\vec{\boldsymbol{\mu}} = NI\vec{\mathbf{A}}} \tag{8.4.8}$$

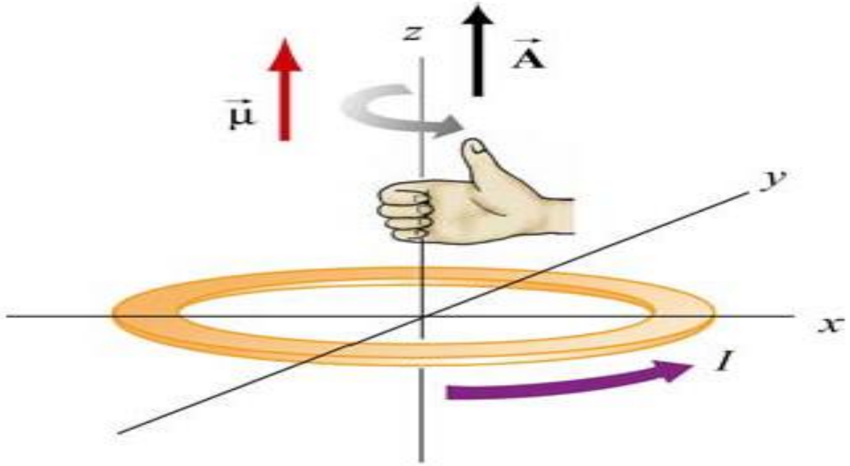


Figure 8.4.3 Right-hand rule for determining the direction of $\vec{\mu}$

The direction of $\vec{\mu}$ is the same as the area vector $\vec{A}$ (perpendicular to the plane of the loop) and is determined by the right-hand rule (Figure 8.4.3). The SI unit for the magnetic dipole moment is ampere-meter^2 $(A \cdot m^2)$. Using the expression for $\vec{\mu}$, the torque exerted on a current-carrying loop can be rewritten as

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}} \qquad (8.4.9)$$

The above equation is analogous to $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$ in Eq. (2.8.3), the torque exerted on an electric dipole moment $\vec{\mathbf{p}}$ in the presence of an electric field $\vec{\mathbf{E}}$. Recalling that the potential energy for an electric dipole is $U = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}$ [see Eq. (2.8.7)], a similar form is expected for the magnetic case. The work done by an external agent to rotate the magnetic dipole from an angle θ_0 to θ is given by

$$\begin{aligned} W_{\text{ext}} &= \int_{\theta_0}^{\theta} \tau \, d\theta' = \int_{\theta_0}^{\theta} (\mu B \sin\theta') \, d\theta' = \mu B \left(\cos\theta_0 - \cos\theta \right) \\ &= \Delta U = U - U_0 \end{aligned} \qquad (8.4.10)$$

Once again, $W_{ext} = -W$, where W is the work done by the magnetic field. Choosing $U_0 = 0$ at $\theta_0 = \pi/2$, the dipole in the presence of an external field then has a potential energy of

$$U = -\mu B\cos\theta = -\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}} \tag{8.4.11}$$

The configuration is at a stable equilibrium when $\vec{\boldsymbol{\mu}}$ is aligned parallel to $\vec{\mathbf{B}}$, making U a minimum with $U_{min} = -\mu B$. On the other hand, when $\vec{\boldsymbol{\mu}}$ and $\vec{\mathbf{B}}$ are anti-parallel, $U_{max} = +\mu B$ is a maximum and the system is unstable.

8.4.1 Magnetic force on a dipole

As we have shown above, the force experienced by a current-carrying rectangular loop (i.e., a magnetic dipole) placed in a uniform magnetic field is zero. What happens if the magnetic field is non-uniform? In this case, there will be a net force acting on the dipole.

Consider the situation where a small dipole $\vec{\mu}$ is placed along the symmetric axis of a bar magnet, as shown in Figure 8.4.4.

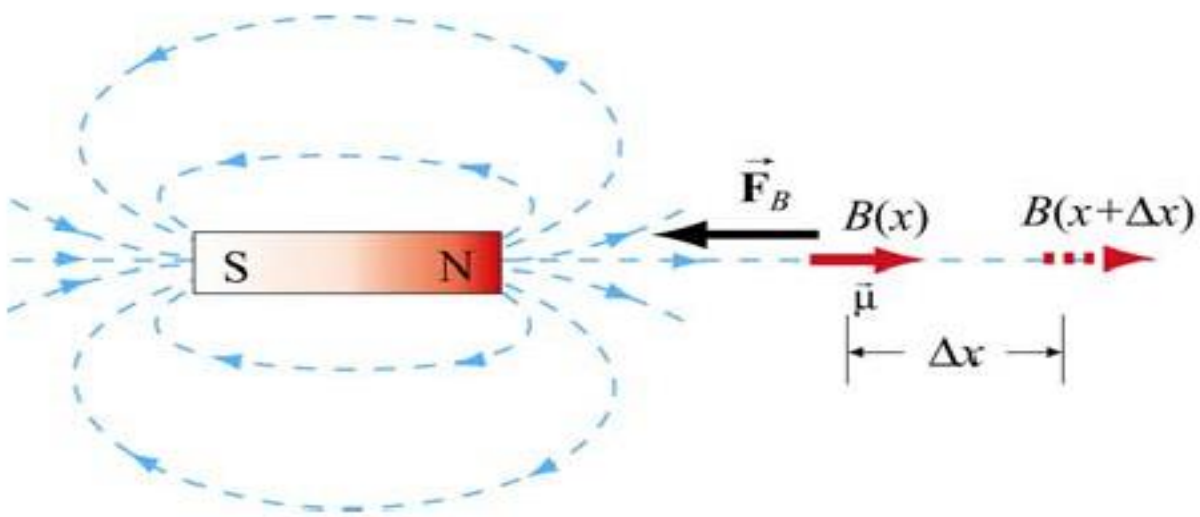


Figure 8.4.4 A magnetic dipole near a bar magnet.

The dipole experiences an attractive force by the bar magnet whose magnetic field is non-uniform in space. Thus, an external force must be applied to move the dipole to the right. The amount of force F_{ext} exerted by an external agent to move the dipole by a distance Δx is given by

$$F_{ext}\Delta x = W_{ext} = \Delta U = -\mu B(x+\Delta x) + \mu B(x) = -\mu[B(x+\Delta x) - B(x)] \quad (8.4.12)$$

where we have used Eq. (8.4.11). For small Δx, the external force may be obtained as

$$F_{\text{ext}} = -\mu \frac{[B(x+\Delta x) - B(x)]}{\Delta x} = -\mu \frac{dB}{dx} \tag{8.4.13}$$

which is a positive quantity since $dB/dx < 0$, i.e., the magnetic field decreases with increasing x. This is precisely the force needed to overcome the attractive force due to the bar magnet. Thus, we have

$$F_B = \mu \frac{dB}{dx} = \frac{d}{dx}(\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}) \tag{8.4.14}$$

More generally, the magnetic force experienced by a dipole $\vec{\boldsymbol{\mu}}$ placed in a non-uniform magnetic field $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}) \tag{8.4.15}$$

where

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{8.4.16}$$

is the gradient operator.