Electricity and Magnetics I

Lecture No.(6)- Semester 1

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The following steps may be useful when applying Gauss's law:

(1) Identify the symmetry associated with the charge distribution.

(2) Determine the direction of the electric field, and a "Gaussian surface" on which the magnitude of the electric field is constant over portions of the surface.

(3) Divide the space into different regions associated with the charge distribution. For each region, calculate q_{enc}, the charge enclosed by the Gaussian surface.

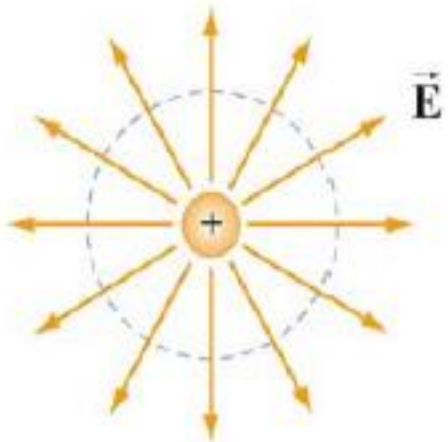
(4) Calculate the electric flux Φ_E through the Gaussian surface for each region.

(5) Equate Φ_E with $q_{\text{enc}} / \varepsilon_0$, and deduce the magnitude of the electric field.

Example 4.1: Infinitely Long Rod of Uniform Charge Density

An infinitely long rod of negligible radius has a uniform charge density λ . Calculate the electric field at a distance r from the wire.

Solution:



We shall solve the problem by following the steps outlined above.

(1) An infinitely long rod possesses cylindrical symmetry.

(2) The charge density is uniformly distributed throughout the length, and the electric field $\vec{\mathbf{E}}$ must be point radially away from the symmetry axis of the rod (Figure 4.2.6). The magnitude of the electric field is constant on cylindrical surfaces of radius r . Therefore, we choose a coaxial cylinder as our Gaussian surface.

(3) The amount of charge enclosed by the Gaussian surface, a cylinder of radius r and length ℓ (Figure 4.2.7), is $q_{enc} = \lambda\ell$.

(4) As indicated in Figure 4.2.7, the Gaussian surface consists of three parts: a two ends S_1 and S_2 plus the curved side wall S_3. The flux through the Gaussian surface is

$$\Phi_E = \oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3$$
$$= 0 + 0 + E_3 A_3 = E\left(2\pi r \ell\right) \qquad (4.2.15)$$

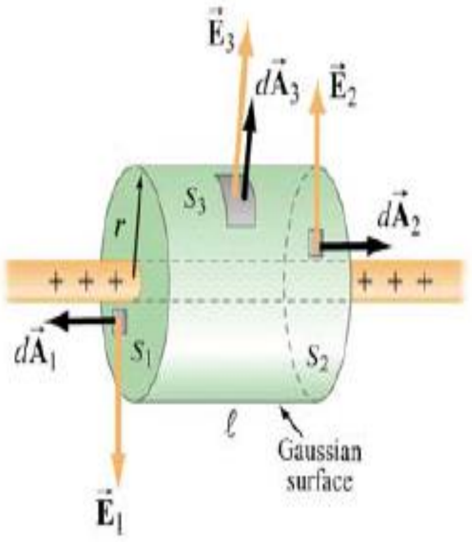


Figure 4.2.7 Gaussian surface for a uniformly charged rod.

where we have set $E_3 = E$. As can be seen from the figure, no flux passes through the ends since the area vectors $d\vec{\mathbf{A}}_1$ and $d\vec{\mathbf{A}}_2$ are perpendicular to the electric field which points in the radial direction.

(5) Applying Gauss's law gives $E\left(2\pi r \ell\right) = \lambda \ell / \varepsilon_0$, or

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad (4.2.16)$$

Example 4.2: Infinite Plane of Charge

Consider an infinitely large non-conducting plane in the *xy*-plane with uniform surface charge density σ. Determine the electric field everywhere in space.

Solution:

(1) An infinitely large plane possesses a planar symmetry.

(2) Since the charge is uniformly distributed on the surface, the electric field $\vec{\mathbf{E}}$ must point perpendicularly away from the plane, $\vec{\mathbf{E}} = E\,\hat{\mathbf{k}}$. The magnitude of the electric field is constant on planes parallel to the non-conducting plane.

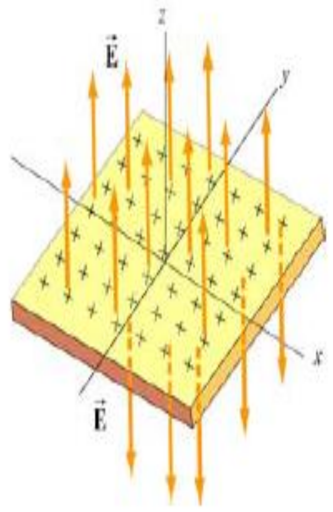
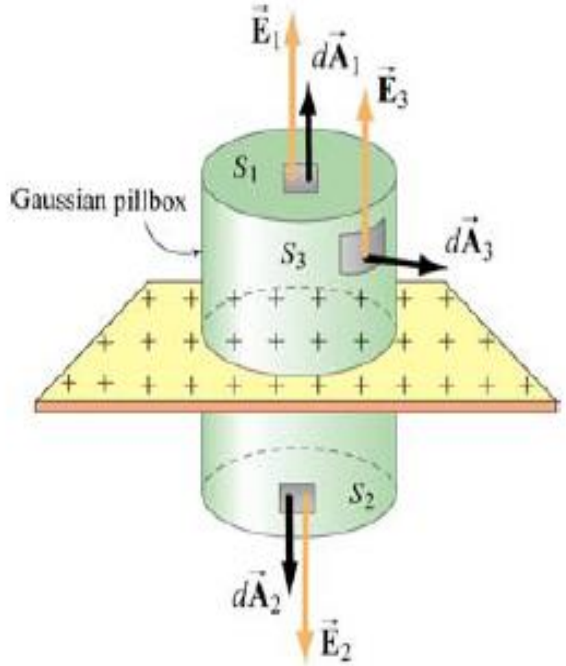


Figure 4.2.9 Electric field for uniform plane of charge

We choose our Gaussian surface to be a cylinder, which is often referred to as a "pillbox" (Figure 4.2.10). The pillbox also consists of three parts: two end-caps S_1 and S_2, and a curved side S_3.



(3) Since the surface charge distribution on is uniform, the charge enclosed by the Gaussian "pillbox" is $q_{\text{enc}} = \sigma A$, where $A = A_1 = A_2$ is the area of the end-caps.

(4) The total flux through the Gaussian pillbox flux is

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \\ &= E_1 A_1 + E_2 A_2 + 0 \\ &= (E_1 + E_2) A\end{aligned} \tag{4.2.17}$$

Since the two ends are at the same distance from the plane, by symmetry, the magnitude of the electric field must be the same: $E_1 = E_2 = E$. Hence, the total flux can be rewritten as

$$\Phi_E = 2EA \tag{4.2.18}$$

(5) By applying Gauss's law, we obtain

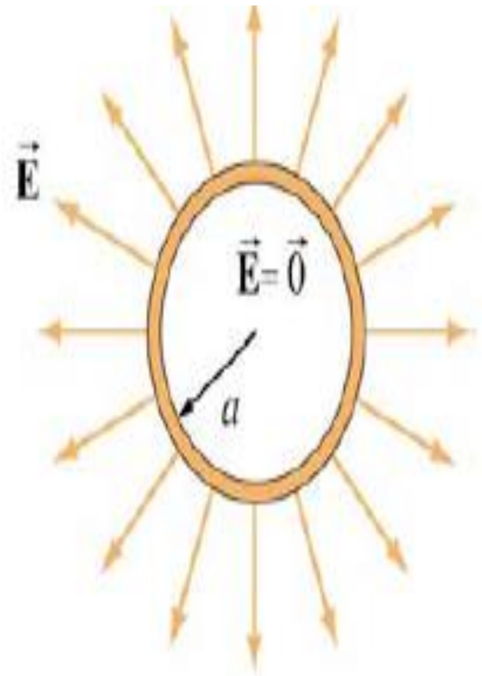
$$2EA = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

which gives

$$E = \frac{\sigma}{2\varepsilon_0} \tag{4.2.19}$$

Example 4.3: Spherical Shell

A thin spherical shell of radius a has a charge $+Q$ evenly distributed over its surface. Find the electric field both inside and outside the shell.



Solutions:

The charge distribution is spherically symmetric, with a surface charge density $\sigma = Q/A_s = Q/4\pi a^2$, where $A_s = 4\pi a^2$ is the surface area of the sphere. The electric field $\vec{\mathbf{E}}$ must be radially symmetric and directed outward (Figure 4.2.12). We treat the regions $r \le a$ and $r \ge a$ separately.

Case 1: $r \le a$

We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 4.2.13(a).

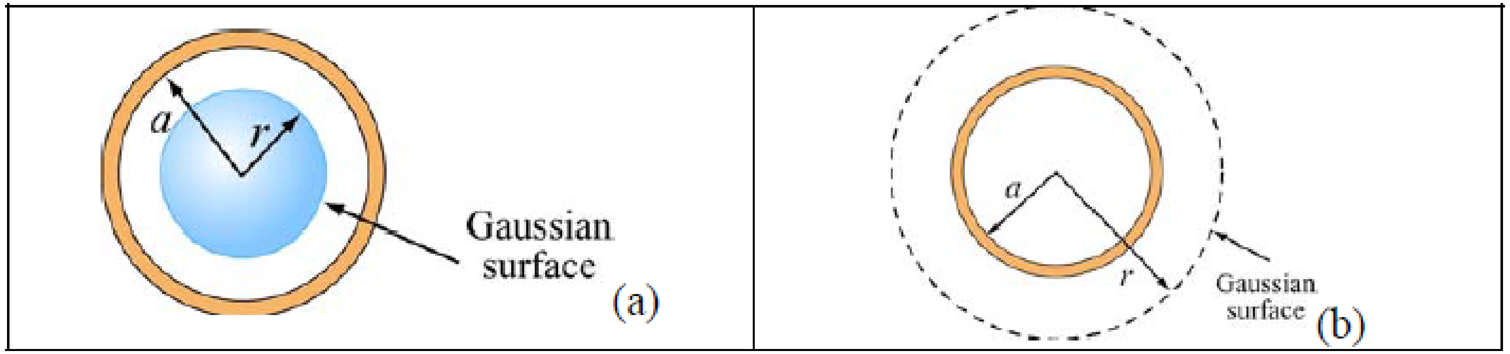


Figure 4.2.13 Gaussian surface for uniformly charged spherical shell for (a) $r < a$, and (b) $r \ge a$

The charge enclosed by the Gaussian surface is $q_{\text{enc}} = 0$ since all the charge is located on the surface of the shell. Thus, from Gauss's law, $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we conclude

$$E = 0, \quad r < a \tag{4.2.22}$$

Case 2: $r \geq a$

In this case, the Gaussian surface is a sphere of radius $r \geq a$, as shown in Figure 4.2.13(b). Since the radius of the "Gaussian sphere" is greater than the radius of the spherical shell, all the charge is enclosed:

$$q_{\text{enc}} = Q$$

Since the flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2)$$

by applying Gauss's law, we obtain

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r \geq a \tag{4.2.23}$$

As in the case of a non-conducting charged plane, we again see a discontinuity in E as we cross the boundary at $r = a$. The change, from outer to the inner surface, is given by

$$\Delta E = E_+ - E_- = \frac{Q}{4\pi\varepsilon_0 a^2} - 0 = \frac{\sigma}{\varepsilon_0}$$

Example 4.4: Non-Conducting Solid Sphere

An electric charge $+Q$ is uniformly distributed throughout a non-conducting solid sphere of radius a. Determine the electric field everywhere inside and outside the sphere.

Solution:

The charge distribution is spherically symmetric with the charge density given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi a^3} \qquad (4.2.24)$$

where V is the volume of the sphere. In this case, the electric field $\vec{\mathbf{E}}$ is radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius r. The regions $r \le a$ and $r \ge a$ shall be studied separately.

Case 1: $r \leq a$.

We choose our Gaussian surface to be a sphere of radius $r \leq a$, as shown in Figure 4.2.15(a).

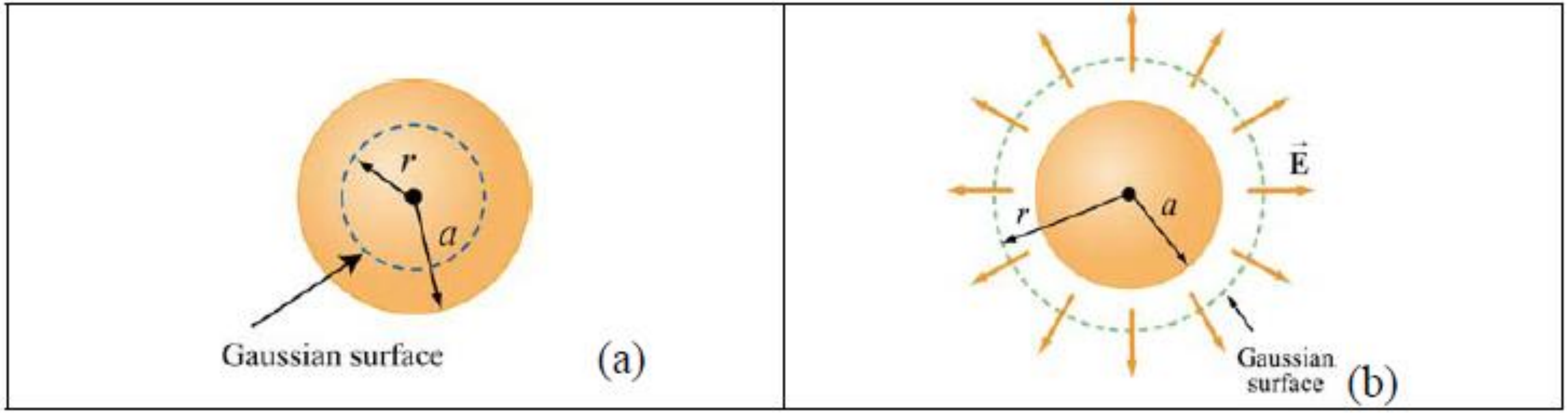


Figure 4.2.15 Gaussian surface for uniformly charged solid sphere, for (a) $r \leq a$, and (b) $r > a$.

The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2)$$

With uniform charge distribution, the charge enclosed is

$$q_{\text{enc}} = \int_V \rho\, dV = \rho V = \rho\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{r^3}{a^3}\right) \tag{4.2.25}$$

which is proportional to the volume enclosed by the Gaussian surface. Applying Gauss's law $\Phi_E = q_{\text{enc}} / \varepsilon_0$, we obtain

$$E\left(4\pi r^2\right) = \frac{\rho}{\varepsilon_0}\left(\frac{4}{3}\pi r^3\right)$$

or

$$\boxed{E = \frac{\rho r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 a^3}, \quad r \le a} \tag{4.2.26}$$

Case 2: $r \geq a$.

In this case, our Gaussian surface is a sphere of radius $r \geq a$, as shown in Figure 4.2.15(b). Since the radius of the Gaussian surface is greater than the radius of the sphere all the charge is enclosed in our Gaussian surface: $q_{\text{enc}} = Q$. With the electric flux through the Gaussian surface given by $\Phi_E = E(4\pi r^2)$, upon applying Gauss's law, we obtain $E(4\pi r^2) = Q / \varepsilon_0$, or

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2} = k_e \frac{Q}{r^2}, \quad r > a} \tag{4.2.27}$$

4.3 Conductors

An insulator such as glass or paper is a material in which electrons are attached to some particular atoms and cannot move freely. On the other hand, inside a conductor, electrons are free to move around. The basic properties of a conductor are the following:

(1) The electric field is zero inside a conductor.

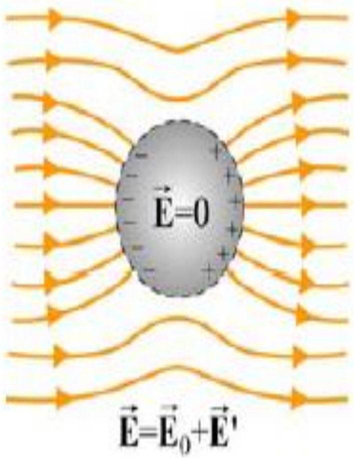


Figure 4.3.1 Placing a conductor in a uniform electric field $\vec{E}_0$.

(2) Any net charge must reside on the surface.

If there were a net charge inside the conductor, then by Gauss's law (Eq. 4.3.2), $\vec{E}$ would no longer be zero there. Therefore, all the net excess charge must flow to the surface of the conductor.

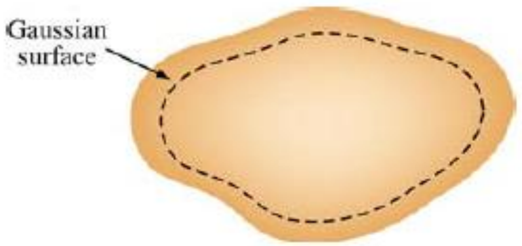


Figure 4.3.2 Gaussian surface inside a conductor. The enclosed charge is zero.

(3) The tangential component of $\vec{\mathbf{E}}$ is zero on the surface of a conductor.

We have already seen that for an isolated conductor, the electric field is zero in its interior. Any excess charge placed on the conductor must then distribute itself on the surface, as implied by Gauss's law.

Consider the line integral $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ around a closed path shown in Figure 4.3.3:

Since the electric field $\vec{\mathbf{E}}$ is conservative, the line integral around the closed path *abcda* vanishes:

$$\oint_{abcda} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_t(\Delta l) - E_n(\Delta x') + 0(\Delta l') + E_n(\Delta x) = 0$$

$$\boxed{E_t = 0 \ \ \text{(on the surface of a conductor)}}$$

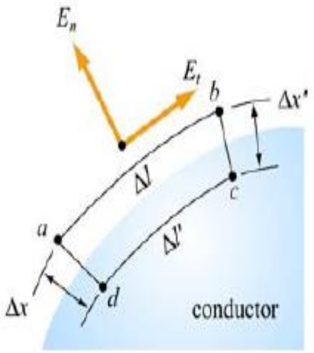


Figure 4.3.3 Normal and tangential components of electric field outside the conductor

(4) $\vec{\mathbf{E}}$ is normal to the surface just outside the conductor.

If the tangential component of $\vec{\mathbf{E}}$ is initially non-zero, charges will then move around until it vanishes. Hence, only the normal component survives.

To compute the field strength just outside the conductor, consider the Gaussian pillbox drawn in Figure 4.3.3. Using Gauss's law, we obtain

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_n A + (0) \cdot A = \frac{\sigma A}{\varepsilon_0} \tag{4.3.2}$$

or

$$\boxed{E_n = \frac{\sigma}{\varepsilon_0}} \tag{4.3.3}$$

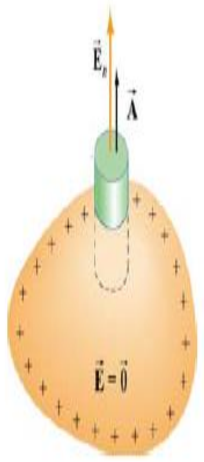


Figure 4.3.3 Gaussian "pillbox" for computing the electric field outside the conductor.

The above result holds for a conductor of arbitrary shape. The pattern of the electric field line directions for the region near a conductor is shown in Figure 4.3.4.

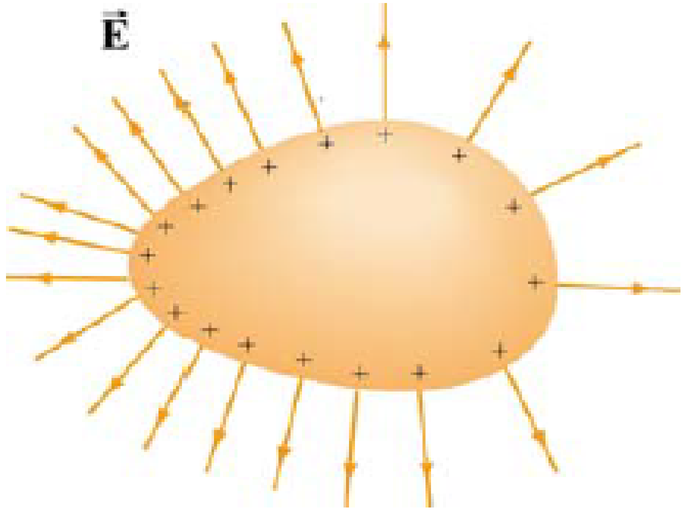


Figure 4.3.4 Just outside the conductor, $\vec{E}$ is always perpendicular to the surface.

As in the examples of an infinitely large non-conducting plane and a spherical shell, the normal component of the electric field exhibits a discontinuity at the boundary:

$$\Delta E_n = E_n^{(+)} - E_n^{(-)} = \frac{\sigma}{\varepsilon_0} - 0 = \frac{\sigma}{\varepsilon_0}$$

Example 4.5: Conductor with Charge Inside a Cavity

Consider a hollow conductor shown in Figure 4.3.5 below. Suppose the net charge carried by the conductor is $+Q$. In addition, there is a charge q inside the cavity. What is the charge on the outer surface of the conductor?

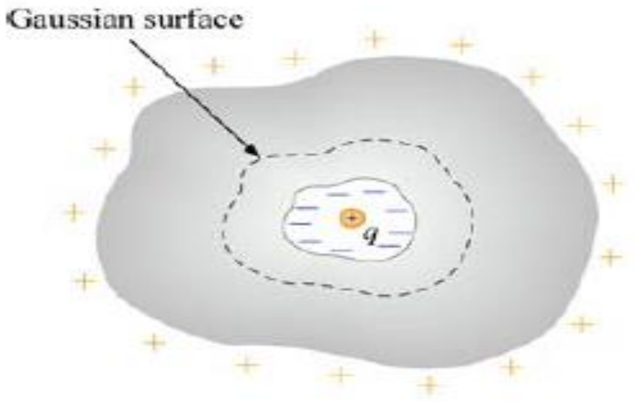


Figure 4.3.5 Conductor with a cavity

Since the electric field inside a conductor must be zero, the net charge enclosed by the Gaussian surface shown in Figure 4.3.5 must be zero. This implies that a charge $-q$ must have been induced on the cavity surface. Since the conductor itself has a charge $+Q$, the amount of charge on the outer surface of the conductor must be $Q + q$.

Example 4.6: Electric Potential Due to a Spherical Shell

Consider a metallic spherical shell of radius a and charge Q, as shown in Figure 4.3.6.

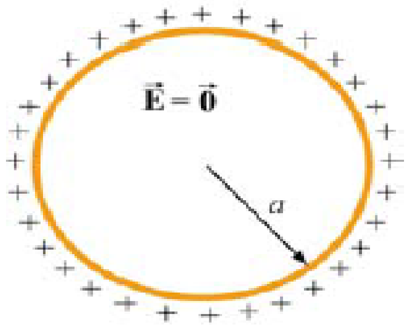


Figure 4.3.6 A spherical shell of radius a and charge Q.

(a) Find the electric potential everywhere.

(b) Calculate the potential energy of the system.

Solution:

(a) In Example 4.3, we showed that the electric field for a spherical shell of is given by

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ 0, & r < a \end{cases}$$

The electric potential may be calculated by using Eq. (3.1.9):

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

For $r > a$, we have

$$V(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2} dr' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = k_e \frac{Q}{r} \tag{4.3.4}$$

where we have chosen $V(\infty) = 0$ as our reference point. On the other hand, for $r < a$, the potential becomes

$$\begin{aligned} V(r) - V(\infty) &= -\int_\infty^a dr E(r > a) - \int_a^r E(r < a) \\ &= -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} = k_e \frac{Q}{a} \end{aligned} \tag{4.3.5}$$

(b) The potential energy U can be thought of as the work that needs to be done to build up the system. To charge up the sphere, an external agent must bring charge from infinity and deposit it onto the surface of the sphere.

Suppose the charge accumulated on the sphere at some instant is q. The potential at the surface of the sphere is then $V = q / 4\pi\varepsilon_0 a$. The amount of work that must be done by an external agent to bring charge dq from infinity and deposit it on the sphere is

$$dW_{\text{ext}} = V\,dq = \left(\frac{q}{4\pi\varepsilon_0 a}\right)dq \tag{4.3.6}$$

Therefore, the total amount of work needed to charge the sphere to Q is

$$W_{\text{ext}} = \int_0^Q dq\,\frac{q}{4\pi\varepsilon_0 a} = \frac{Q^2}{8\pi\varepsilon_0 a} \tag{4.3.7}$$

Since $V = Q/4\pi\varepsilon_0 a$ and $W_{\text{ext}} = U$, the above expression is simplified to

$$\boxed{U = \frac{1}{2}QV} \tag{4.3.8}$$

The result can be contrasted with the case of a point charge. The work required to bring a point charge Q from infinity to a point where the electric potential due to other charges is V would be $W_{\text{ext}} = QV$. Therefore, for a point charge Q, the potential energy is $U=QV$.

Now, suppose two metal spheres with radii r_1 and r_2 are connected by a thin conducting wire, as shown in Figure 4.3.8.

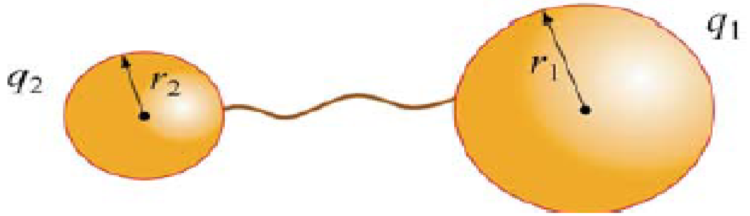


Figure 4.3.8 Two conducting spheres connected by a wire.

Charge will continue to flow until equilibrium is established such that both spheres are at the same potential $V_1 = V_2 = V$. Suppose the charges on the spheres at equilibrium are q_1 and q_2. Neglecting the effect of the wire that connects the two spheres, the equipotential condition implies

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}$$

or

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} \qquad (4.3.9)$$

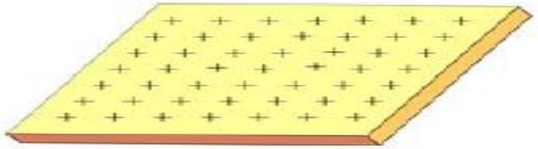
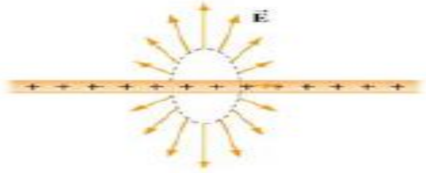
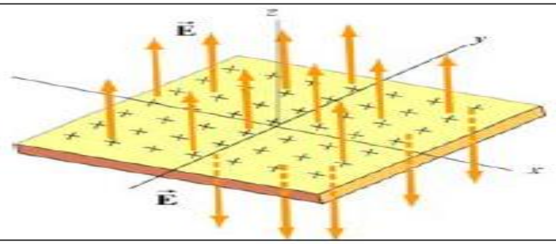
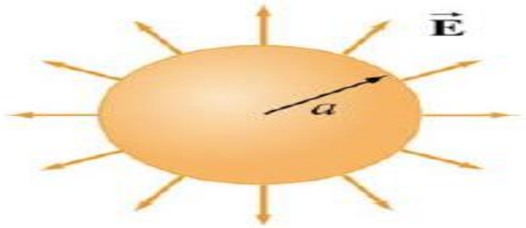
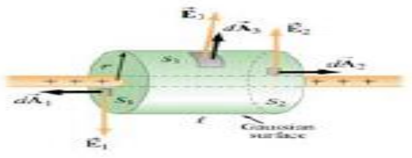
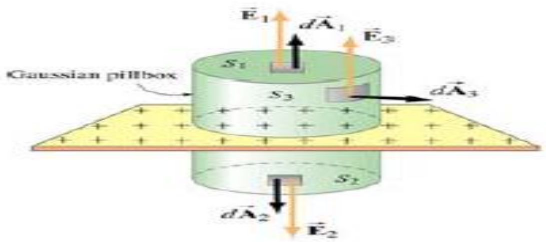
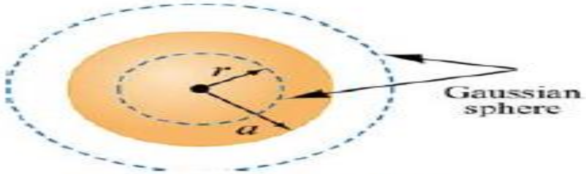
assuming that the two spheres are very far apart so that the charge distributions on the surfaces of the conductors are uniform. The electric fields can be expressed as

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} = \frac{\sigma_1}{\varepsilon_0}, \qquad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} = \frac{\sigma_2}{\varepsilon_0} \tag{4.3.10}$$

where σ_1 and σ_2 are the surface charge densities on spheres 1 and 2, respectively. The two equations can be combined to yield

$$\boxed{\frac{E_1}{E_2} = \frac{\sigma_1}{\sigma_2} = \frac{r_2}{r_1}} \tag{4.3.11}$$

With the surface charge density being inversely proportional to the radius, we conclude that the regions with the smallest radii of curvature have the greatest σ. Thus, the electric field strength on the surface of a conductor is greatest at the sharpest point. The design of a lightning rod is based on this principle.

System	Infinite line of charge	Infinite plane of charge	Uniformly charged solid sphere
Figure			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{E}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$
Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{\text{enc}} = \lambda l$	$q_{\text{enc}} = \sigma A$	$q_{\text{enc}} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{\text{in}} / \varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$

4.8.2 Electric Flux Through a Square Surface

(a) Compute the electric flux through a square surface of edges $2l$ due to a charge $+Q$ located at a perpendicular distance l from the center of the square, as shown in Figure 4.8.4.

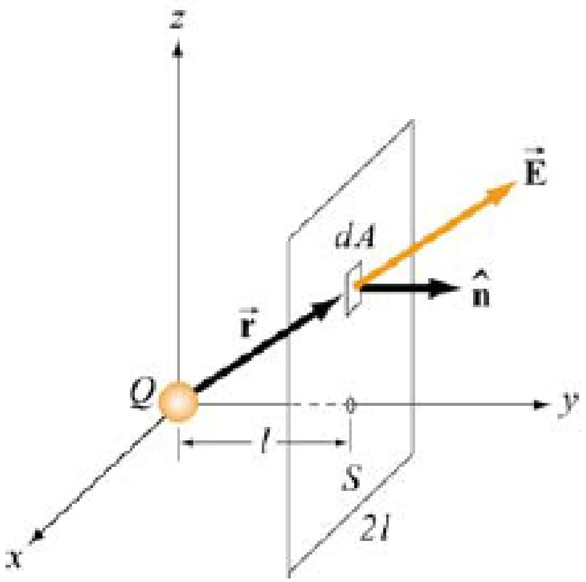


Figure 4.8.4 Electric flux through a square surface

Solutions:

(a) The electric field due to the charge $+Q$ is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right)$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ in Cartesian coordinates. On the surface S, $y = l$ and the area element is $d\vec{\mathbf{A}} = dA\hat{\mathbf{j}} = (dx\,dz)\hat{\mathbf{j}}$. Since $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$ and $\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1$, we have

$$\vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right)\cdot(dx\,dz)\hat{\mathbf{j}} = \frac{Ql}{4\pi\varepsilon_0 r^3}\,dx\,dz$$

Thus, the electric flux through S is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Ql}{4\pi\varepsilon_0} \int_{-l}^{l} dx \int_{-l}^{l} \frac{dz}{(x^2 + l^2 + z^2)^{3/2}} = \frac{Ql}{4\pi\varepsilon_0} \int_{-l}^{l} dx \left. \frac{z}{(x^2 + l^2)(x^2 + l^2 + z^2)^{1/2}} \right|_{-l}^{l}$$

$$= \frac{Ql}{2\pi\varepsilon_0} \int_{-l}^{l} \frac{l\,dx}{(x^2 + l^2)(x^2 + 2l^2)^{1/2}} = \frac{Q}{2\pi\varepsilon_0} \tan^{-1} \left. \left(\frac{x}{\sqrt{x^2 + 2l^2}} \right) \right|_{-l}^{l}$$

$$= \frac{Q}{2\pi\varepsilon_0} \left[\tan^{-1}(1/\sqrt{3}) - \tan^{-1}(-1/\sqrt{3}) \right] = \frac{Q}{6\varepsilon_0}$$

where the following integrals have been used:

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 (x^2 + a^2)^{1/2}}$$

$$\int \frac{dx}{(x^2 + a^2)(x^2 + b^2)^{1/2}} = \frac{1}{a(b^2 - a^2)^{1/2}} \tan^{-1} \sqrt{\frac{b^2 - a^2}{a^2 (x^2 + b^2)}}, \quad b^2 > a^2$$

(b) Using the result obtained in (a), if the charge $+Q$ is now at the center of a cube of side $2l$ (Figure 4.8.5), what is the total flux emerging from all the six faces of the closed surface?

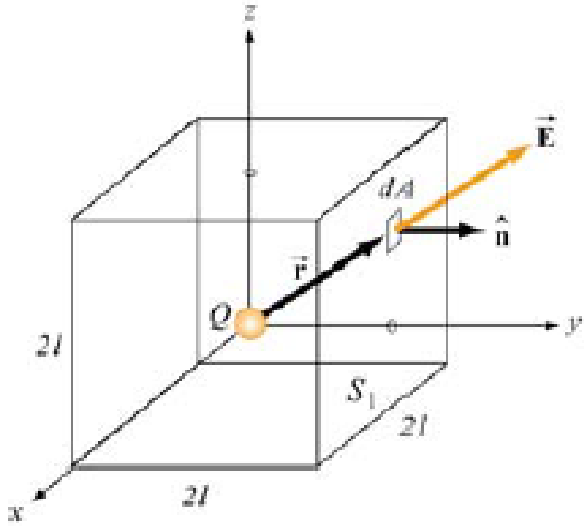


Figure 4.8.5 Electric flux through the surface of a cube

(b) From symmetry arguments, the flux through each face must be the same. Thus, the total flux through the cube is just six times that through one face:

$$\Phi_E = 6\left(\frac{Q}{6\varepsilon_0}\right) = \frac{Q}{\varepsilon_0}$$

The result shows that the electric flux Φ_E passing through a closed surface is proportional to the charge enclosed. In addition, the result further reinforces the notion that Φ_E is independent of the shape of the closed surface.

4.8.4 Electric Potential of a Uniformly Charged Sphere

An insulated solid sphere of radius a has a uniform charge density ρ. Compute the electric potential everywhere.

Solution: Using Gauss's law, we showed in Example 4.4 that the electric field due to the charge distribution is

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ \dfrac{Qr}{4\pi\varepsilon_0 a^3}\hat{\mathbf{r}}, & r < a \end{cases} \tag{4.8.3}$$

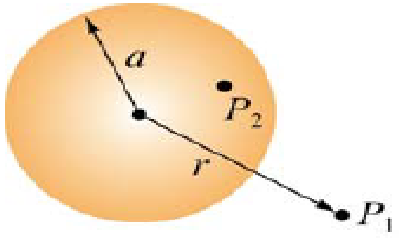


Figure 4.8.6

The electric potential at P_1 (indicated in Figure 4.8.6) outside the sphere is

$$V_1(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = k_e\frac{Q}{r} \tag{4.8.4}$$

On the other hand, the electric potential at P_2 inside the sphere is given by

$$V_2(r) - V(\infty) = -\int_{\infty}^{a} dr E(r > a) - \int_{a}^{r} E(r < a) = -\int_{\infty}^{a} dr \frac{Q}{4\pi\varepsilon_0 r^2} - \int_{a}^{r} dr' \frac{Qr}{4\pi\varepsilon_0 a^3} r'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}\frac{1}{2}\left(r^2 - a^2\right) = \frac{1}{8\pi\varepsilon_0}\frac{Q}{a}\left(3 - \frac{r^2}{a^2}\right) \qquad (4.8.5)$$

$$= k_e \frac{Q}{2a}\left(3 - \frac{r^2}{a^2}\right)$$