Electricity and Magnetics I

Lecture No.(9)- Semester 1

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2017-2018

Current and Resistance

Electric currents are flows of electric charge. Suppose a collection of charges is moving perpendicular to a surface of area A, as shown in Figure 6.1.1.

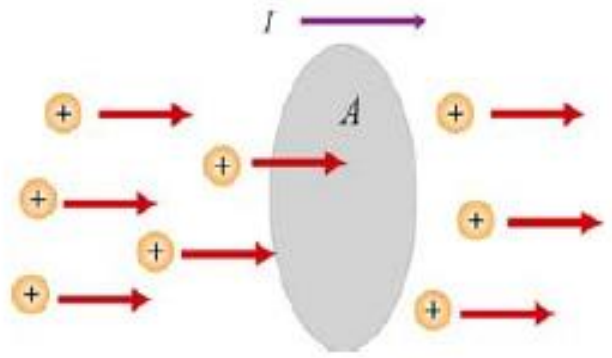

Figure 6.1.1 Charges moving through a cross section.

The electric current is defined to be the rate at which charges flow across any cross-sectional area. If an amount of charge ΔQ passes through a surface in a time interval Δt, then the average current I_{avg} is given by

$$I_{avg} = \frac{\Delta Q}{\Delta t} \qquad (6.1.1)$$

The SI unit of current is the ampere (A), with 1 A = 1 coulomb/sec.

$$I = \frac{dQ}{dt} \qquad (6.1.2)$$

Since flow has a direction, we have implicitly introduced a convention that the direction of current corresponds to the direction in which positive charges are flowing. The flowing charges inside wires are negatively charged electrons that move in the opposite direction of the current. Electric currents flow in conductors: solids (metals, semiconductors), liquids (electrolytes, ionized) and gases (ionized), but the flow is impeded in non-conductors or insulators.

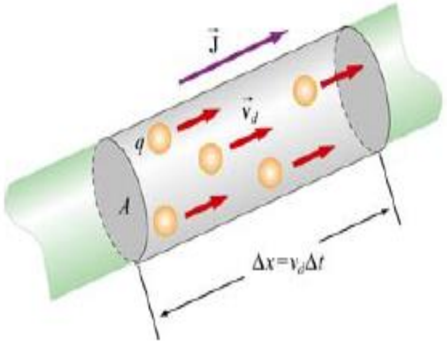

Figure 6.1.2 A microscopic picture of current flowing in a conductor.

$$I = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} \tag{6.1.3}$$

where $\vec{\mathbf{J}}$ is the current density (the SI unit of current density are $\mathrm{A/m^2}$). If q is the charge of each carrier, and n is the number of charge carriers per unit volume, the total amount of charge in this section is then $\Delta Q = q(nA\,\Delta x)$. Suppose that the charge carriers move with a speed v_d; then the displacement in a time interval Δt will be $\Delta x = v_d \Delta t$, which implies

$$I_{\text{avg}} = \frac{\Delta Q}{\Delta t} = nqv_d A \tag{6.1.4}$$

The speed v_d at which the charge carriers are moving is known as the *drift speed*. Physically, v_d is the average speed of the charge carriers inside a conductor when an external electric field is applied. Actually an electron inside the conductor does not travel in a straight line; instead, its path is rather erratic, as shown in Figure 6.1.3.

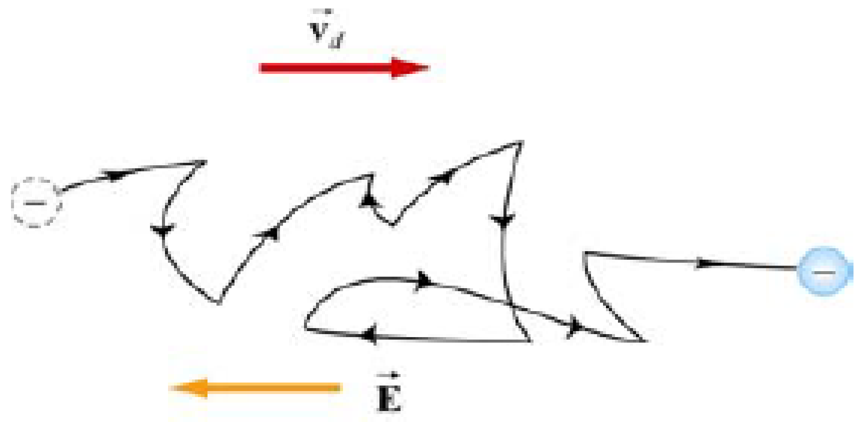

Figure 6.1.3 Motion of an electron in a conductor.

From the above equations, the current density $\vec{\mathbf{J}}$ can be written as

$$\vec{\mathbf{J}} = nq\vec{\mathbf{v}}_d \qquad (6.1.5)$$

Thus, we see that $\vec{\mathbf{J}}$ and $\vec{\mathbf{v}}_d$ point in the same direction for positive charge carriers, in opposite directions for negative charge carriers.

To find the drift velocity of the electrons, we first note that an electron in the conductor experiences an electric force $\vec{\mathbf{F}}_e = -e\vec{\mathbf{E}}$ which gives an acceleration

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}_e}{m_e} = -\frac{e\vec{\mathbf{E}}}{m_e} \qquad (6.1.6)$$

Let the velocity of a given electron immediate after a collision be $\vec{\mathbf{v}}_i$. The velocity of the electron immediately before the next collision is then given by

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}\,t = \vec{\mathbf{v}}_i - \frac{e\vec{\mathbf{E}}}{m_e}t \tag{6.1.7}$$

where t is the time traveled. The average of $\vec{\mathbf{v}}_f$ over all time intervals is

$$\langle \vec{\mathbf{v}}_f \rangle = \langle \vec{\mathbf{v}}_i \rangle - \frac{e\vec{\mathbf{E}}}{m_e}\langle t \rangle \tag{6.1.8}$$

which is equal to the drift velocity $\vec{\mathbf{v}}_d$. Since in the absence of electric field, the velocity of the electron is completely random, it follows that $\langle \vec{\mathbf{v}}_i \rangle = 0$. If $\tau = \langle t \rangle$ is the average characteristic time between successive collisions (the *mean free time*), we have

$$\vec{\mathbf{v}}_d = \langle \vec{\mathbf{v}}_f \rangle = -\frac{e\vec{\mathbf{E}}}{m_e}\tau \qquad (6.1.9)$$

The current density in Eq. (6.1.5) becomes

$$\vec{\mathbf{J}} = -ne\vec{\mathbf{v}}_d = -ne\left(-\frac{e\vec{\mathbf{E}}}{m_e}\tau\right) = \frac{ne^2\tau}{m_e}\vec{\mathbf{E}} \qquad (6.1.10)$$

Note that $\vec{\mathbf{J}}$ and $\vec{\mathbf{E}}$ will be in the same direction for either negative or positive charge carriers.

6.2 Ohm's Law

In many materials, the current density is linearly dependent on the external electric field $\vec{\mathbf{E}}$. Their relation is usually expressed as

$$\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \tag{6.2.1}$$

where σ is called the *conductivity* of the material. The above equation is known as the (microscopic) Ohm's law. A material that obeys this relation is said to be ohmic; otherwise, the material is non-ohmic.

Comparing Eq. (6.2.1) with Eq. (6.1.10), we see that the conductivity can be expressed as

$$\sigma = \frac{ne^2\tau}{m_e} \tag{6.2.2}$$

To obtain a more useful form of Ohm's law for practical applications, consider a segment of straight wire of length l and cross-sectional area A, as shown in Figure 6.2.1.

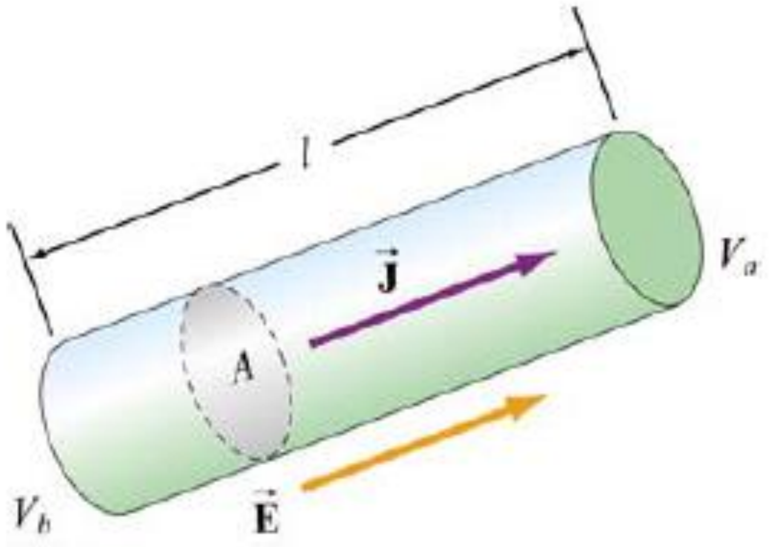

Figure 6.2.1 A uniform conductor of length l and potential difference $\Delta V = V_b - V_a$.

Suppose a potential difference $\Delta V = V_b - V_a$ is applied between the ends of the wire, creating an electric field $\vec{\mathbf{E}}$ and a current I. Assuming $\vec{\mathbf{E}}$ to be uniform, we then have

$$\Delta V = V_b - V_a = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = El \qquad (6.2.3)$$

The current density can then be written as

$$J = \sigma E = \sigma \left(\frac{\Delta V}{l} \right) \qquad (6.2.4)$$

With $J = I / A$, the potential difference becomes

$$\Delta V = \frac{l}{\sigma} J = \left(\frac{l}{\sigma A} \right) I = RI \qquad (6.2.5)$$

where

$$R = \frac{\Delta V}{I} = \frac{l}{\sigma A} \qquad (6.2.6)$$

is the resistance of the conductor. The equation

$$\Delta V = IR \qquad (6.2.7)$$

is the "macroscopic" version of the Ohm's law. The SI unit of R is the ohm (Ω, Greek letter Omega), where

$$1\,\Omega \equiv \frac{1\text{ V}}{1\text{ A}} \qquad (6.2.8)$$

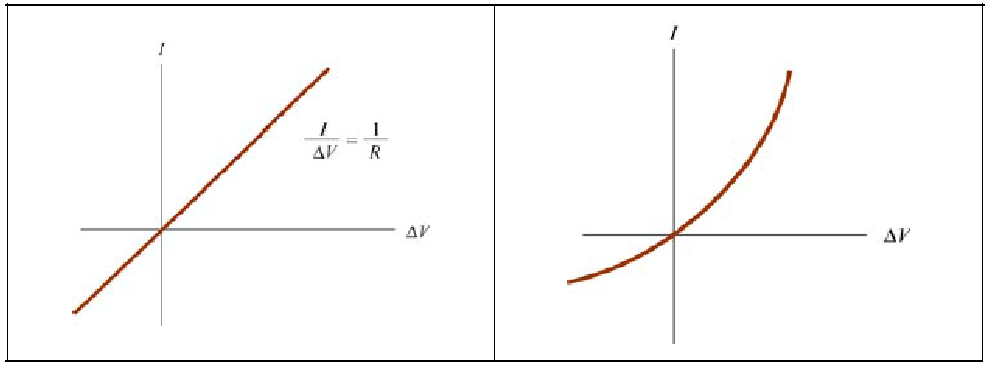

Figure 6.2.2 Ohmic vs. Non-ohmic behavior.

The resistivity ρ of a material is defined as the reciprocal of conductivity,

$$\rho = \frac{1}{\sigma} = \frac{m_e}{ne^2\tau} \qquad (6.2.9)$$

From the above equations, we see that ρ can be related to the resistance R of an object by

$$\rho = \frac{E}{J} = \frac{\Delta V\ /\ l}{I\ /\ A} = \frac{RA}{l}$$

or

$$R = \frac{\rho l}{A} \qquad (6.2.10)$$

The resistivity of a material actually varies with temperature *T*. For metals, the variation is linear over a large range of *T*:

$$\rho = \rho_0 \left[1 + \alpha(T - T_0)\right] \qquad (6.2.11)$$

where α is the *temperature coefficient of resistivity*. Typical values of ρ , σ and α (at $20°C$) for different types of materials are given in the Table below.

Material	Resistivity ρ $(\Omega \cdot m)$	Conductivity σ $(\Omega \cdot m)^{-1}$	Temperature Coefficient α $(°C)^{-1}$
Elements Silver	1.59×10^{-8}	6.29×10^{7}	0.0038
Copper	1.72×10^{-8}	5.81×10^{7}	0.0039
Aluminum	2.82×10^{-8}	3.55×10^{7}	0.0039
Tungsten	5.6×10^{-8}	1.8×10^{7}	0.0045
Iron	10.0×10^{-8}	1.0×10^{7}	0.0050
Platinum	10.6×10^{-8}	1.0×10^{7}	0.0039
Alloys Brass	7×10^{-8}	1.4×10^{7}	0.002
Manganin	44×10^{-8}	0.23×10^{7}	1.0×10^{-5}
Nichrome	100×10^{-8}	0.1×10^{7}	0.0004
Semiconductors Carbon (graphite)	3.5×10^{-5}	2.9×10^{4}	–0.0005
Germanium (pure)	0.46	2.2	–0.048
Silicon (pure)	640	1.6×10^{-3}	–0.075
Insulators Glass	$10^{10}-10^{14}$	$10^{-14}-10^{-10}$	
Sulfur	10^{15}	10^{-15}	
Quartz (fused)	7.5×10^{16}	1.33×10^{-18}	

6.3 Electrical Energy and Power

Consider a circuit consisting of a battery and a resistor with resistance R (Figure 6.3.1). Let the potential difference between two points a and b be $\Delta V = V_b - V_a > 0$. If a charge Δq is moved from a through the battery, its electric potential energy is increased by $\Delta U = \Delta q \, \Delta V$. On the other hand, as the charge moves across the resistor, the potential energy is decreased due to collisions with atoms in the resistor. If we neglect the internal resistance of the battery and the connecting wires, upon returning to a the potential energy of Δq remains unchanged.

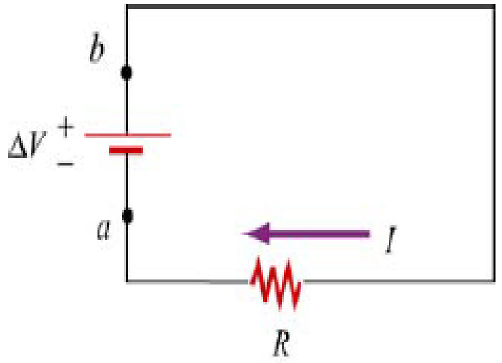

Figure 6.3.1 A circuit consisting of a battery and a resistor of resistance *R*.

Thus, the rate of energy loss through the resistor is given by

$$P = \frac{\Delta U}{\Delta t} = \left(\frac{\Delta q}{\Delta t} \right) \Delta V = I \Delta V \qquad (6.3.1)$$

This is precisely the power supplied by the battery. Using $\Delta V = IR$, one may rewrite the above equation as

$$\boxed{P = I^2 R = \frac{(\Delta V)^2}{R}} \qquad (6.3.2)$$

- The **electric current** is defined as:

$$I = \frac{dQ}{dt}$$

- The average current in a conductor is

$$I_{avg} = nqv_d A$$

where n is the number density of the charge carriers, q is the charge each carrier has, v_d is the **drift speed**, and A is the cross-sectional area.

- The **current density** J through the cross sectional area of the wire is

$$\vec{\mathbf{J}} = nq\vec{\mathbf{v}}_d$$

- Microscopic **Ohm's law**: the current density is proportional to the electric field, and the constant of proportionality is called **conductivity** σ :

$$\vec{\mathbf{J}} = \sigma \vec{\mathbf{E}}$$

- The reciprocal of conductivity σ is called **resistivity** ρ :

$$\rho = \frac{1}{\sigma}$$

- Macroscopic Ohm's law: The **resistance** R of a conductor is the ratio of the potential difference ΔV between the two ends of the conductor and the current I:

$$R = \frac{\Delta V}{I}$$

- Resistance is related to resistivity by

$$R = \frac{\rho l}{A}$$

where l is the length and A is the cross-sectional area of the conductor.

- The **drift velocity** of an electron in the conductor is

$$\vec{\mathbf{v}}_d = -\frac{e\vec{\mathbf{E}}}{m_e}\tau$$

where m_e is the mass of an electron, and τ is the average time between successive collisions.

- The resistivity of a metal is related to τ by

$$\rho = \frac{1}{\sigma} = \frac{m_e}{ne^2\tau}$$

- The temperature variation of resistivity of a conductor is

$$\rho = \rho_0\left[1+\alpha\left(T-T_0\right)\right]$$

where α is the **temperature coefficient of resistivity**.

- **Power**, or rate at which energy is delivered to the resistor is

$$P = I\Delta V = I^2 R = \frac{(\Delta V)^2}{R}$$

6.5.1 Resistivity of a Cable

A 3000-km long cable consists of seven copper wires, each of diameter 0.73 mm, bundled together and surrounded by an insulating sheath. Calculate the resistance of the cable. Use $3\times10^{-6}\,\Omega\cdot\text{cm}$ for the resistivity of the copper.

Solution:

The resistance R of a conductor is related to the resistivity ρ by $R=\rho l/A$, where l and A are the length of the conductor and the cross-sectional area, respectively. Since the cable consists of $N=7$ copper wires, the total cross sectional area is

$$A = N\pi r^2 = N\frac{\pi d^2}{4} = 7\frac{\pi(0.073\,\text{cm})^2}{4}$$

The resistance then becomes

$$R = \frac{\rho l}{A} = \frac{\left(3\times10^{-6}\,\Omega\cdot\text{cm}\right)\left(3\times10^{8}\,\text{cm}\right)}{7\pi\left(0.073\text{cm}\right)^{2}/4} = 3.1\times10^{4}\,\Omega$$

6.5.2 Charge at a Junction

Show that the total amount of charge at the junction of the two materials in Figure 6.5.1 is $\varepsilon_0 I(\sigma_2^{-1} - \sigma_1^{-1})$, where I is the current flowing through the junction, and σ_1 and σ_2 are the conductivities for the two materials.

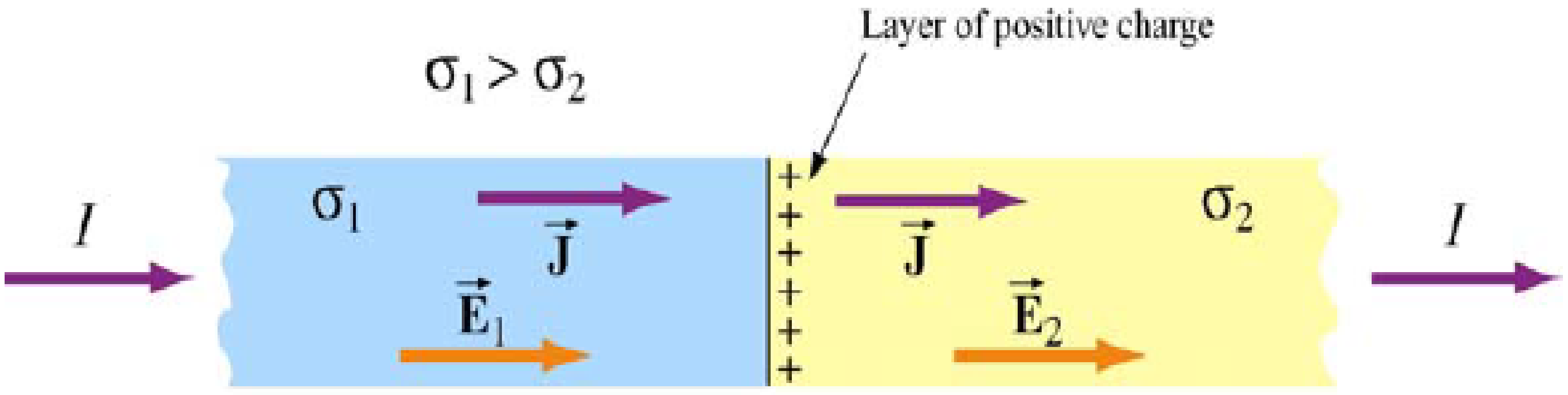

Figure 6.5.1 Charge at a junction.

Solution:

In a steady state of current flow, the normal component of the current density $\vec{\mathbf{J}}$ must be the same on both sides of the junction. Since $J = \sigma E$, we have $\sigma_1 E_1 = \sigma_2 E_2$

or

$$E_2 = \left(\frac{\sigma_1}{\sigma_2}\right) E_1$$

Let the charge on the interface be q_{in}, we have, from the Gauss's law:

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = (E_2 - E_1) A = \frac{q_{\text{in}}}{\varepsilon_0}$$

or

$$E_2 - E_1 = \frac{q_{\text{in}}}{A\varepsilon_0}$$

Substituting the expression for E_2 from above then yields

$$q_{\text{in}} = \varepsilon_0 A E_1 \left(\frac{\sigma_1}{\sigma_2} - 1 \right) = \varepsilon_0 A \sigma_1 E_1 \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

Since the current is $I = JA = (\sigma_1 E_1) A$, the amount of charge on the interface becomes

$$q_{\text{in}} = \varepsilon_0 I \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right)$$

6.5.3 Drift Velocity

The resistivity of seawater is about $25\ \Omega\cdot\text{cm}$. The charge carriers are chiefly Na^{+} and Cl^{-} ions, and of each there are about $3\times10^{20}/\text{cm}^3$. If we fill a plastic tube 2 meters long with seawater and connect a 12-volt battery to the electrodes at each end, what is the resulting average drift velocity of the ions, in cm/s?

Solution:

The current in a conductor of cross sectional area A is related to the drift speed v_d of the charge carriers by

$$I = enAv_d$$

where n is the number of charges per unit volume. We can then rewrite the Ohm's law as

$$V = IR = (neAv_d)\left(\frac{\rho l}{A}\right) = nev_d\rho l$$

which yields

$$v_d = \frac{V}{ne\rho l}$$

Substituting the values, we have

$$v_d = \frac{12\text{V}}{\left(6\times10^{20}/\text{cm}^3\right)\left(1.6\times10^{-19}\,\text{C}\right)\left(25\Omega\cdot\text{cm}\right)\left(200\text{cm}\right)} = 2.5\times10^{-5}\,\frac{\text{V}\cdot\text{cm}}{\text{C}\cdot\Omega} = 2.5\times10^{-5}\,\frac{\text{cm}}{\text{s}}$$

In converting the units we have used

$$\frac{\text{V}}{\Omega\cdot\text{C}} = \left(\frac{\text{V}}{\Omega}\right)\frac{1}{\text{C}} = \frac{\text{A}}{\text{C}} = \text{s}^{-1}$$

6.5.5 Resistance of a Hollow Cylinder

Consider a hollow cylinder of length L and inner radius a and outer radius b, as shown in Figure 6.5.3. The material has resistivity ρ.

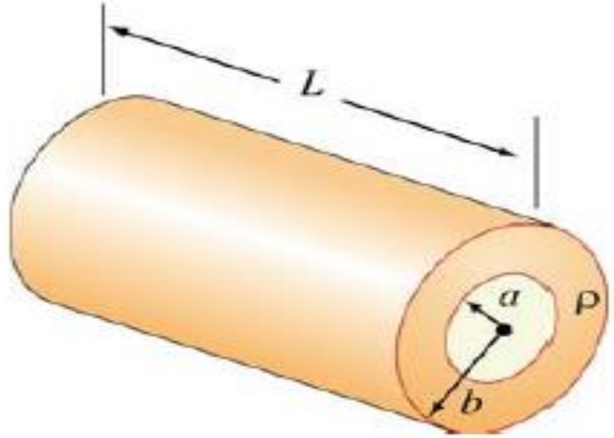

Figure 6.5.3 A hollow cylinder.

(a) Suppose a potential difference is applied between the ends of the cylinder and produces a current flowing parallel to the axis. What is the resistance measured?

(b) If instead the potential difference is applied between the inner and outer surfaces so that current flows radially outward, what is the resistance measured?

Solution:

(a) When a potential difference is applied between the ends of the cylinder, current flows parallel to the axis. In this case, the cross-sectional area is $A = \pi(b^2 - a^2)$, and the resistance is given by

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi(b^2 - a^2)}$$

(b) Consider a differential element which is made up of a thin cylinder of inner radius r and outer radius $r + dr$ and length L. Its contribution to the resistance of the system is given by

$$dR = \frac{\rho\, dl}{A} = \frac{\rho\, dr}{2\pi r L}$$

where $A = 2\pi r L$ is the area normal to the direction of current flow. The total resistance of the system becomes

$$R = \int_a^b \frac{\rho\, dr}{2\pi r L} = \frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)$$

Homework

Q1/ A sphere of radius 10 mm that carries a charge of 8 nC $= 8\times10^{-9}$ C is whirled in a circle at the end of an insulated string. The rotation frequency is 100π rad/s.

(a) What is the basic definition of current in terms of charge?

(b) What average current does this rotating charge represent?

(c) What is the average current density over the area traversed by the sphere?

Q2/ A 1500 W radiant heater is constructed to operate at 115 V.

(a) What will be the current in the heater? [Ans. ~10 A]

(b) What is the resistance of the heating coil? [Ans. ~10 Ω]

(c) How many kilocalories are generated in one hour by the heater? (1 Calorie = 4.18 J)

Q3/ A wire with a resistance of 6.0 Ω is drawn out through a die so that its new length is three times its original length. Find the resistance of the longer wire, assuming that the resistivity and density of the material are not changed during the drawing process. [Ans: 54 Ω].

Q4/ A 100-W light bulb is plugged into a standard 120-V outlet. (a) How much does it cost per month (31 days) to leave the light turned on? Assume electricity costs 6 cents per kW·h. (b) What is the resistance of the bulb? (c) What is the current in the bulb? [Ans: (a) $4.46; (b) 144 Ω; (c) 0.833 A].

Q5/ Figure 6.7.3 shows a three-layer sandwich made of two resistive materials with resistivities ρ_1 and ρ_2. From left to right, we have a layer of material with resistivity ρ_1 of width $d/3$, followed by a layer of material with resistivity ρ_2, also of width $d/3$, followed by another layer of the first material with resistivity ρ_1, again of width $d/3$.

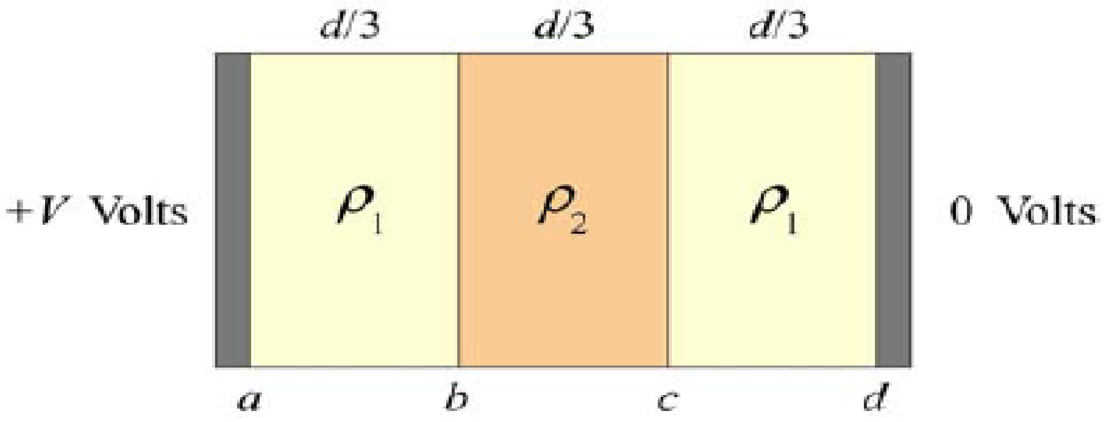

Figure 6.7.3 Charge accumulation at interface.

There are four interfaces between the various materials and the conductors, which we label a through d, as indicated on the sketch. A steady current I flows through this sandwich from left to right, corresponding to a current density $J = I/A$.

(a) What are the electric fields $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$ in the two different dielectric materials? To obtain these fields, assume that the current density is the same in every layer. Why must this be true? [Ans: All fields point to the right, $E_1 = \rho_1 I/A$, $E_2 = \rho_2 I/A$; the current densities must be the same in a steady state, otherwise there would be a continuous buildup of charge at the interfaces to unlimited values.]

(b) What is the total resistance R of this sandwich? Show that your expression reduces to the expected result if $\rho_1 = \rho_2 = \rho$. [Ans: $R = d(2\rho_1 + \rho_2)/3A$; if $\rho_1 = \rho_2 = \rho$, then $R = d\,\rho / A$, as expected.]

(c) As we move from right to left, what are the changes in potential across the three layers, in terms of V and the resistivities? [Ans: $V\rho_1/(2\rho_1 + \rho_2)$, $V\rho_2/(2\rho_1 + \rho_2)$, $V\rho_1/(2\rho_1 + \rho_2)$, summing to a total potential drop of V, as required].

(d) What are the charges per unit area, σ_a through σ_d, at the interfaces? Use Gauss's Law and assume that the electric field in the conducting caps is zero. [Ans: $\sigma_a = -\sigma_d = 3\varepsilon_0 V \rho_1 / d(2\rho_1 + \rho_2)$, $\sigma_b = -\sigma_c = 3\varepsilon_0 V(\rho_2 - \rho_1)/d(2\rho_1 + \rho_2)$.]

(e) Consider the limit $\rho_2 \gg \rho_1$. What do your answers above reduce to in this limit?