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تفاضل وتكامل

II

الفصل الدراسي الثاني

٢٠١٦ - ٢٠١٧

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تفاضل وتكامل II

Improper integrals

$\int_a^b f(x)\,dx$ exists if $f(x)$ is piceswise continuous in the closed interval $[a,b]$. However, in many interesting application, one of two situations occurs, either

(1) a or b is infinite or

(2) f becomes infinite at one or more values in the interval $[a,b]$. If one of these cases occurs, we say that the integral is an improper integral.

<u>Case 1</u>: $\boxed{b=+\infty, \; a=-\infty, \text{ or both}}$

<u>Def^n(1)</u>: (i) Let a be a real number and let f be a function having the property that $\int_a^N f(x)\,dx$ exists for every real number $N \geq a$. Then we define the improper integral

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx \quad \text{—— ①}$$

provided that this limit exists.

If $\int_a^\infty f(x)\,dx$ exists and is finite, we say that the improper integral is convergent. If the limit in ① does not exist, or if it exists and is infinite, then we say that the improper integral is divergent.

(ii) If $\int_{-N}^b f(x)\,dx$ exists for every real N such that $-N \leq b$, we define $\int_{-\infty}^b f(x)\,dx = \lim_{N\to\infty} \int_{-N}^b f(x)\,dx$

whenever the limit exists. we define the terms convergent and divergent an in (i).

(iii) If $\int_{-M}^{0} f(x)\,dx$ and $\int_{0}^{N} f(x)\,dx$ exist for every real N and M, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty}\int_{0}^{N} f(x)\,dx + \lim_{M\to\infty}\int_{-M}^{0} f(x)\,dx$$

whenever both these limits exist.

Ex: (1) Evaluate $\int_{1}^{\infty}\frac{dx}{x}$

$$\int_{1}^{\infty}\frac{dx}{x} = \lim_{N\to\infty}\int_{1}^{N}\frac{dx}{x}$$

$$\int_{1}^{N}\frac{dx}{x} = \ln x\Big|_{1}^{N} = \ln N - \ln 1 = \ln N, \text{ but}$$

$$\lim_{N\to\infty} \ln x\Big|_{1}^{N} = \lim_{N\to\infty}\ln N = \infty$$

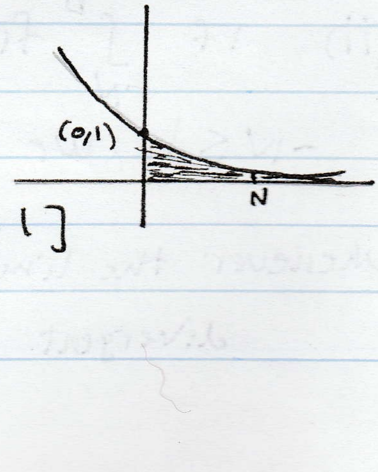
So the Improper Integral is divergent.

(2) $\int_{-\infty}^{0} e^{x}\,dx$; $\int_{-\infty}^{0} e^{x}\,dx = \lim_{N\to\infty}\int_{-N}^{0} e^{x}\,dx = \lim_{N\to\infty} e^{x}\Big|_{-N}^{0}$

$$= \lim_{N\to\infty}\left[1 - e^{-N}\right] = \lim_{N\to\infty}\left[1 - \frac{1}{e^{N}}\right] = 1$$

$\int_{-\infty}^{0} e^{x}\,dx$ Convergent

(3) Calculate the area in the first quadrant under the curve $y = e^{-x}$

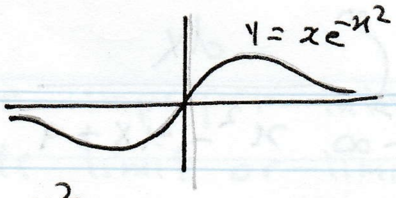


Area from 0 to N $= A_{0}^{N}$

$$= \lim_{N\to\infty}\int_{0}^{N} e^{-x}\,dx = \lim_{N\to\infty} -e^{-x}\Big|_{0}^{N} = \lim_{N\to\infty}\left[-e^{-N}+1\right]$$

$$= 0 + 1 = 1$$

4) Calculate $\int_{-\infty}^{\infty} x e^{-x^2} dx$



$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$$

$$= \lim_{M\to\infty} \int_{-M}^{0} x e^{-x^2} dx + \lim_{N\to\infty} \int_{0}^{N} x e^{-x^2} dx$$

$$= \lim_{M\to\infty} -\frac{1}{2} e^{-x^2}\Big|_{-M}^{0} + \lim_{N\to 0} -\frac{1}{2} e^{-x^2}\Big|_{0}^{N}$$

$$= \lim_{M\to\infty} \left[-\frac{1}{2} + \frac{1}{e^{-m^2}}\right] + \lim_{N\to 0} \left[-\frac{1}{2} e^{-N^2} + \frac{1}{2}\right]$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

5) Evaluate $\int_{1}^{\infty} \left(\frac{1}{x^K}\right) dx$, $K > 0$, $K \neq 1$

$$\int_{1}^{\infty} \frac{1}{x^K} dx = \lim_{N\to\infty} \int_{1}^{N} \frac{1}{x^K} dx = \lim_{N\to\infty} \int_{1}^{N} x^{-K} dx$$

$$= \lim_{N\to\infty} \frac{x^{-K+1}}{-K+1}\Big|_{1}^{N} = \lim_{N\to\infty} \frac{1}{-K+1}\left[x^{-K+1}\right]_{1}^{N}$$

$$= \frac{1}{-K+1}\left[N^{-K+1} - 1\right] = \frac{1}{K-1}\left(1 - N^{-K+1}\right)$$

If $0 < K < 1$, then $N^{1-K} \to \infty$ an $n \to \infty$ and the integral diverges. If $K > 1$, then $N^{1-K} = \frac{1}{N^{K-1}} \to 0$ as $n \to \infty$, and the integral converges. that is

$$\int_{1}^{\infty} \frac{1}{x^K} dx = \begin{cases} \text{diverges if } 0 < K < 1 \\ \text{converges if } K > 1 \text{ to } \frac{1}{K-1} \end{cases}$$

Calculate   H.W.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-4x+9}$$

Case 2: a and b are finite and f becomes infinite at some number in the closed interval $[a,b]$.

Def^n 2: let a and b be finite numbers.

(i) If $\int_{a+\epsilon}^{b} f(x)\,dx$ exists for every $\epsilon>0$ in $(0,b-a]$ and if f has a vertical asymptote at $\boxed{x=a}$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx \quad \text{—— (2)}$$

provided that the limit exists.

(ii) If $\int_a^{b-\epsilon} f(x)\,dx$ exists for every ϵ in $(0,b-a]$ and if f has a vertical asymptote at $\boxed{x=b}$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx \quad \text{—— (3)}$$

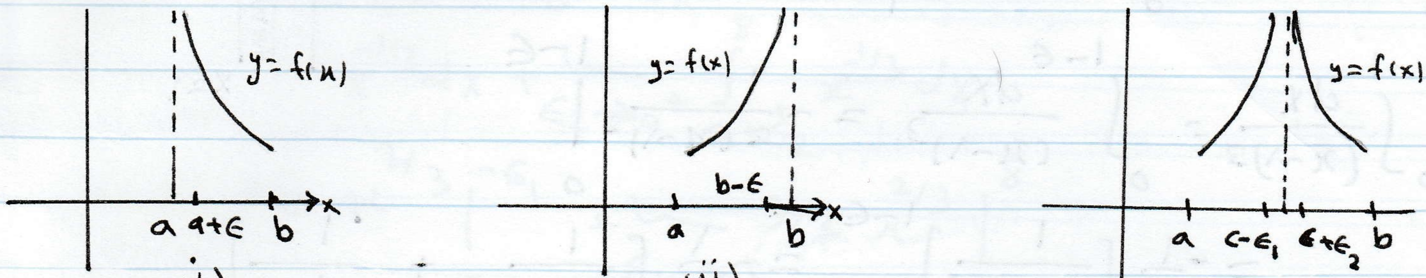
provided that the limit exists.

(iii) If for c in (a,b) f has a vertical asymptote at $\boxed{x=c}$ and if the integrals $\int_a^{c-\epsilon_1} f(x)\,dx$ and $\int_{c+\epsilon_2}^{b} f(x)\,dx$ exist for ϵ_1 in $(0,c-a]$ and ϵ_2 in $(0,b-c]$, then

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1\to 0^+} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0^+} \int_{c+\epsilon_2}^{b} f(x)\,dx \quad \text{—— (4)}$$

provided that both these integrals exist.

For each (i), (ii) and (iii), the improper integral is convergent if the appropriate limit or limits exist and are finite. Otherwise it is divergent.



(i) vertical asym. at left endpoint

(ii) vertical asym. at right endpoint

(iii) vertical asym. in (a,b)

Ex: ① Evaluate $\displaystyle\int_0^1 \frac{dx}{x}$, $x \neq 0$

$[a,b] = [0,1]$, $\quad a+\epsilon = 0+\epsilon = \epsilon$

$$\int_0^1 \frac{dx}{x} = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} \frac{dx}{x} = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{1} \frac{dx}{x} = \lim_{\epsilon\to 0^+} \ln x \Big|_{\epsilon}^{1}$$

$$= \lim_{\epsilon\to 0^+} -\ln\epsilon \Rightarrow \infty \text{ as } \epsilon \to 0^+$$

$\displaystyle\int_0^1 \frac{dx}{x}$ div.

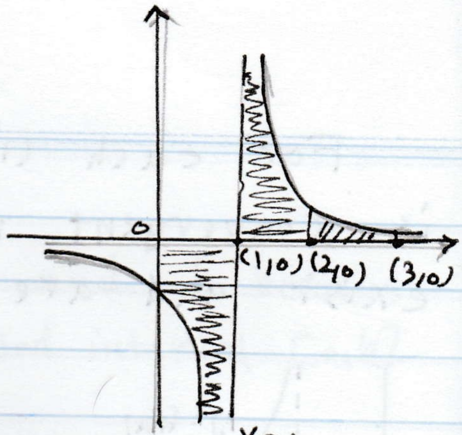
② $\displaystyle\int_0^2 \frac{dx}{(x-2)^{4/5}}$, $x \neq 2$

$$\int_0^2 \frac{dx}{(x-2)^{4/5}} = \lim_{\epsilon\to 0^+} \int_0^{2-\epsilon} \frac{dx}{(x-2)^{4/5}} = \lim_{\epsilon\to 0^+} 5(x-2)^{1/5}\Big|_0^{2-\epsilon}$$

$$= \lim_{\epsilon\to 0^+} 5[(2-\epsilon-2)+2]^{1/5} = 5\lim_{\epsilon\to 0^+}(-\epsilon+2)^{1/5}$$

$$= 5\cdot 2^{1/5} < \infty \quad \text{Conv.}$$

3) Calculate $\int_0^3 \frac{dx}{(x-1)^3}$



$$\int_0^3 \frac{dx}{(x-1)^3} = \int_0^1 \frac{dx}{(x-1)^3} + \int_1^3 \frac{dx}{(x-1)^3}$$

$$\int_0^1 \frac{dx}{(x-1)^3} = \int_0^{1-\epsilon} \frac{dx}{(x-1)^3} = -\frac{1}{2(x-1)^2}\Big|_0^{1-\epsilon}$$

$$= -\frac{1}{2}\left[\frac{1}{(x-1)^2}\right]_0^{1-\epsilon} = -\frac{1}{2}\left[\frac{1}{(1-\epsilon-1)^2} + \frac{1}{1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{\epsilon^2} - 1\right] = \infty, \text{as } \epsilon \to 0^+. \quad \Rightarrow \text{div.}$$

$$\int_1^3 \frac{dx}{(x-1)^3} = -\frac{1}{2}\left[\frac{1}{(x-1)^2}\right|_{1+\epsilon}^{3} = -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{1+\epsilon-1}\right] =$$

$$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{\epsilon^2}\right] \to \infty \text{ as } \epsilon \to 0^+$$

Ex: Calculate $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

Here we have vertical asymptotes at both endpoints.

$$x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$$

$$1 = \sin\theta \Rightarrow \sin^{-1}(1) = \theta \Rightarrow \theta = \frac{\pi}{2}$$

$$-1 = \sin\theta \Rightarrow \theta = -\pi/2$$

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cos\theta\, d\theta}{\cos\theta} = \theta\Big|_{-\frac{\pi}{2}}^{\pi/2} = \pi$$

We have transformed an improper integral into an ordinary one.

H.W.

Ex: calculate $\int_{-1}^{8} \frac{dx}{\sqrt[3]{x}}$, $x \neq 0$

$$\int_{-1}^{8} \frac{dx}{\sqrt[3]{x}} = \int_{-1}^{0} \frac{dx}{\sqrt[3]{x}} + \int_{0}^{8} \frac{dx}{\sqrt[3]{x}}$$

$$= \lim_{\epsilon_1 \to 0^+} \int_{-1}^{0-\epsilon_1} x^{-1/3} dx + \lim_{\epsilon_2 \to 0^+} \int_{0+\epsilon_2}^{8} x^{-1/3} dx$$

$$= \lim_{\epsilon_1 \to 0^+} 3/2 \, x^{2/3} \Big|_{-1}^{-\epsilon_1} + \lim_{\epsilon_2 \to 0^+} 3/2 \, x^{2/3} \Big|_{\epsilon_2}^{8}$$

$$= \lim_{\epsilon_1 \to 0^+} \frac{3}{2} \left[-\epsilon_1^{2/3} + 1 \right] + \lim_{\epsilon_2 \to 0^+} \frac{3}{2} \left[8^{2/3} - \epsilon_2^{2/3} \right]$$

$$= \frac{3}{2} + \frac{3}{2}[4] = \frac{3}{2} + 6 = \frac{15}{2} \quad \text{Conv.}$$

H.W: Determine whether the given improper integral converges or diverges. If it converges. calculate ~~the~~ its value.

$$\int_0^\infty e^{-2x} dx \quad , \quad \int_{-\infty}^{\infty} e^{-0.01x} dx \quad ; \quad \int_{-\infty}^{\infty} x^3 e^{-x^4} dx$$

$$\int_{-\infty}^{1} \frac{dx}{\sqrt{4-x}} \quad , \quad \int_{10}^{\infty} \frac{dx}{x^2-1} \quad , \quad \int_{2}^{3} \frac{dx}{(x-2)^{1/3}}$$

$$\int_{1}^{\infty} \frac{dx}{x \ln x} \quad ; \quad \int_{-\infty}^{0} e^{x} \cos x \, dx \quad , \quad \int_{0}^{5} \frac{dx}{\sqrt{25-x^2}}$$