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تفاضل وتكامل

II

الفصل الدراسي الثاني

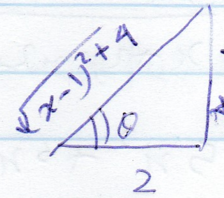
٢٠١٦ – ٢٠١٧

H.W

$$\int \frac{dx}{(x^2-2x+5)^2} = \int \frac{dx}{[(x-1)^2+4]^2} = \int \frac{dx}{(4+(x-1)^2)^2}$$

let $x-1 = 2\tan\theta \Rightarrow \tan\theta = \frac{x-1}{2}$

$dx = 2\sec^2\theta \, d\theta$



$$\int \frac{2\sec^2\theta \, d\theta}{[4+4\tan^2\theta]^2} = \int \frac{2\sec^2\theta \, d\theta}{(4\sec^2\theta)^2} = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{8}\int \cos^2\theta \, d\theta = \frac{1}{8}\int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta$$

$$\sin\theta = \frac{x-1}{\sqrt{(x-1)^2+4}} = \frac{x-1}{\sqrt{x^2-2x+5}}, \quad \cos\theta = \frac{2}{\sqrt{x^2-2x+5}}$$

$$\frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta = \frac{1}{16}\left[\tan^{-1}\frac{x-1}{2} + \frac{x-1}{\sqrt{x^2-2x+5}} \cdot \frac{2}{\sqrt{x^2-2x+5}}\right]$$

$I =$

H.W

Ex: $\int \frac{2x^2-5x+2}{x^3+x} \, dx$

$$\frac{2x^2-5x+2}{x^3+x} = \frac{2x^2-5x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x^2-5x+2 = A(x^2+1) + x(Bx+C) = Ax^2 + A + Bx^2 + Cx$$

$\therefore A+B=2, \quad C=-5, \quad A=2, \quad B=0$

$$\int \frac{2}{x} \, dx + \int \frac{-5}{x^2+1} \, dx = 2\ln|x| - 5\tan^{-1}x + C.$$

$$\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx$$ H.W

$$\frac{5x^2+6x+2}{(x+2)(x^2+2x+5)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+5} = \frac{A(x^2+2x+5)+(Bx+C)(x+2)}{(x+2)(x^2+2x+5)}$$

$$5x^2+6x+2 = A(x^2+2x+5)+(Bx+C)(x+2)$$
$$= Ax^2+2Ax+5A+Bx^2+2Bx+Cx+2C$$

$A+B=5$ $\qquad A+B=5 \Rightarrow A=5-B$

$2A+2B+C=6$ $\quad , 2(5-B)+2B+C=6 \Rightarrow 10-2B+2B+C=6$

$5A+2C=2$ $\qquad \boxed{C=-4}$

$5A=2-2C=2+8=10 \Rightarrow A=\frac{10}{5}=2$

$\boxed{B=3}$ $\qquad \boxed{A=2}$

$$I=\int\frac{2}{x+2}dx+\int\frac{3x-4}{x^2+2x+5}dx = 2\ln|x+2| + \int\frac{3x-4}{x^2+2x+5}dx$$

$$\int\frac{3x-4}{x^2+2x+5}dx = \int\frac{2x+x-4}{x^2+2x+5}dx = \int\frac{2x+2+x-6}{x^2+2x+5}dx$$

$$=\int\frac{2x+2}{x^2+2x+5}dx + \int\frac{x-6}{x^2+2x+5}dx$$

$$=\ln|x^2+2x+5| + \int\frac{x+1-7}{x^2+2x+5}dx$$

$$\int\frac{x-6}{x^2+2x+5}dx = \int\frac{x+1-7}{x^2+2x+5}dx = \int\frac{(x+1)\,dx}{x^2+2x+5} - 7\int\frac{dx}{x^2+2x+5}$$

$$=\frac{1}{2}\ln|x^2+2x+5| - 7\int\frac{dx}{(x+1)^2+4}$$

$$=\frac{1}{2}\ln|x^2+2x+5| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$$

$$I = 2\ln|x+2| + \ln|x^2+2x+5| + \frac{1}{2}\ln|x^2+2x+5| - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

Other Substitutions

When an integrand contains a term of the form $[f(x)]^{m/n}$, then the substitution

$$u = (f(x))^{1/n}$$ will often be useful.

Ex: calculate $\int \frac{dx}{2+\sqrt{x}}$

let $u = x^{1/2} \Rightarrow u^2 = x \Rightarrow dx = 2u\,du$

$$\int \frac{2u\,du}{2+u} = \int \left(2 - \frac{4}{u+2}\right) du = 2u - 4\ln|u+2|$$

$$= 2\sqrt{x} - 4\ln|\sqrt{x}+2| + c$$

Ex: $\int \frac{(x-1)}{6\sqrt{x}(4+\sqrt[3]{x})}\, dy$ $\qquad u = x^{1/6} \Rightarrow u^6 = x$

$dx = 6u^5\,du$

$$\int \frac{(x-1)\,dx}{6\sqrt{x}(4+\sqrt[3]{x})} = \int \frac{(u^6-1)\,6u^5\,du}{6u^3(4+u^2)} = \int \frac{u^8-u^2}{u^2+4}\,du$$

$$= \int \left(u^6 - 4u^4 + 16u^2 - 65 + \frac{260}{u^2+4}\right) du$$

$$= \frac{u^7}{7} - \frac{4}{5}u^5 + \frac{16}{3}u^3 - 65u + 130\tan^{-1}\frac{u}{2}$$

$$= \frac{x^{7/6}}{7} - \frac{4}{5}x^{5/6} + \frac{16}{3}x^{1/2} - 65x^{1/6} + 130\tan^{-1}\left(\frac{x^{1/6}}{2}\right) + c$$

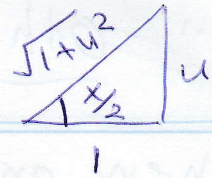
[illegible]

Note: Certain rational functions of $\sin x$ & $\cos x$ ~~tanx~~, can be integrated with the use ~~of~~ of a special Trigonometric substitution. [illegible]

$$u = \tan\frac{x}{2}$$

$u = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} u$

$\Rightarrow x = 2\tan^{-1} u$



$$\boxed{dx = \frac{2}{1+u^2}\, du}$$

$\sin\frac{x}{2} = \frac{u}{\sqrt{1+u^2}}$, $\cos\frac{x}{2} = \frac{1}{\sqrt{1+u^2}}$

$\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= 2\frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$

$\boxed{\sin x = \frac{2u}{1+u^2}}$, $\boxed{\cos x = \frac{1-u^2}{1+u^2}}$

$\boxed{\tan x = \frac{2u}{1-u^2}}$

Ex: $\int \frac{dx}{2+\cos x}$, let $u = \tan\frac{x}{2}$

$\tan^{-1} u = \frac{x}{2} \Rightarrow 2\tan^{-1} u = x$

$\Rightarrow dx = \frac{2}{1+u^2}\, du$

$$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2\,du}{1+u^2}}{2+\frac{1-u^2}{1+u^2}} = \int \frac{2\,du}{1+u^2} \cdot \frac{1+u^2}{3+u^2}$$

$$= \int \frac{2\,du}{3+u^2} = 2\int \frac{du}{3+u^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{\tan x/2}{\sqrt{3}}\right] + C$$

Solve: $\int \frac{dx}{\sin x - \cos x}$, $\int \frac{\cot x\, dx}{1-\cos x}$, $\int \frac{dx}{1+\sin x+\cos x}$

$\int \frac{dx}{\sin x + \cos x}$

Ex: calculate

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} \quad ; \quad u = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} u$$

$$\Rightarrow x = 2\tan^{-1} u \Rightarrow dx = \frac{2\,du}{1+u^2}$$

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x} = \int_0^1 \frac{\frac{2\,du}{1+u^2}}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} = \int_0^1 \frac{2\,du}{1+2u-u^2}$$

$$= \int_0^1 \frac{2\,du}{1-(u^2-2u)} = \int_0^1 \frac{2\,du}{1-(u^2-2u+1-1)} = \int_0^1 \frac{2\,du}{2-(u-1)^2}$$

let $(u-1) = \sqrt{2}\sin\theta \Rightarrow u = \sqrt{2}\sin\theta - 1$

$\Rightarrow du = \sqrt{2}\cos\theta\, d\theta$, $\theta = 0$, when $u = 1$

and $\sin\theta = \dfrac{u-1}{\sqrt{2}}$, $\theta = \sin^{-1}\left(\dfrac{u-1}{\sqrt{2}}\right)$

$u = 0 \Rightarrow \theta = \sin^{-1}\left(-\dfrac{1}{\sqrt{2}}\right) = -\dfrac{\pi}{4}$

$$\int_0^1 \frac{2\,du}{2-(u-1)^2} = \int_{-\frac{\pi}{4}}^{0} \frac{2\sqrt{2}\cos\theta\, d\theta}{2-(2\sin^2\theta)} = \int_{-\frac{\pi}{4}} \frac{\sqrt{2}\cos\theta\, d\theta}{1-\sin^2\theta}$$

$$= \int_{-\frac{\pi}{4}}^{0} \frac{\sqrt{2}\cos\theta\, d\theta}{\cos^2\theta} = \int_{-\frac{\pi}{4}}^{0} \sqrt{2}\sec\theta\, d\theta$$

$$= \sqrt{2}\ln|\sec\theta + \tan\theta|\Big|_{-\frac{\pi}{4}}^{0} =$$

$$= \sqrt{2}\left[\ln 1 - \ln(\sqrt{2}-1)\right]$$

$$\simeq 1.25$$