محاضرات

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تفاضل وتكامل

II

الفصل الدراسي الثاني

٢٠١٦ - ٢٠١٧

Techniques of Integration

1- Integration by parts.

$d(uv) = u\,dv + v\,du$, integrating both sides,

$uv = \int u\,dv + \int v\,du$

$$\int u\,dv = uv - \int v\,du$$

Ex: ① Calculate $\int x e^{x} dx$

let $u = x \Rightarrow du = dx$

$dv = e^{x} dx \Rightarrow v = \int e^{x} dx = e^{x}$

$\int x e^{x} dx = x e^{x} - \int e^{x} dx = x e^{x} - e^{x} + C.$

② $\int_0^{\pi/2} x \cos x \, dx$; $u = x \Rightarrow du = dx$

$dv = \cos x \, dx \Rightarrow v = \int \cos x \, dx = \sin x$

$\int_0^{\pi/2} x \cos x \, dx = x \sin x \Big|_0^{\pi/2} - \int \sin x \, dx = x \sin x + \cos x \Big]_0^{\pi/2}$

$= \frac{\pi}{2} - 1$

③ $\int \ln x \, dx$, $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = dx \Rightarrow v = x$

$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + C$

Note: In many of the integrations involving $\ln x$, we may take $u = \ln x$ so that $du = \frac{1}{x} dx$ and the ln term vanishes.

4) $\int_0^1 x^3 e^{x^2} dx$, $u = x^2 \Rightarrow du = 2x\,dx$; $dv = \int x e^{x^2} dx$

$= \frac{1}{2} e^{x^2}$

$\int x^3 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} \int e^{x^2} 2x \, dx$

$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} \Big]_0^1 = \frac{1}{2} e - \frac{1}{2} e + \frac{1}{2} = \frac{1}{2}$

Ex: $\int e^{x} \sin x \, dx$

$I = \int e^{x} \sin x \, dx, \quad u = e^{x} \Rightarrow du = e^{x} dx$

$dv = \sin x \, dx \Rightarrow v = -\cos x$

$I = -e^{x} \cos x + \int \cos x \, e^{x} dx$

$\int \cos x \, e^{x} dx, \quad u = e^{x} \Rightarrow du = e^{x} dx, \; v = \int \cos x \, dx = \sin x$

$\int \cos x \, e^{x} dx = e^{x} \sin x - \int \sin x \, e^{x} dx = e^{x} \sin x - I$

$I = -e^{x} \cos x + e^{x} \sin x - I$

$2I = -e^{x} \cos x + e^{x} \sin x = e^{x} (\sin x - \cos x)$

$I = \frac{1}{2} e^{x} (\sin x - \cos x) + C$

Ex: $\int \tan^{-1} x \, dx, \quad u = \tan^{-1} x \Rightarrow du = \frac{dx}{1+x^2}$

$v = \int dx = x$

$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \frac{dx}{1+x^2}$

$= x \tan^{-1} x - \int \frac{x \, dx}{1+x^2} = x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + C$

H.W: calculate the integral:

1- $\int \sin^2 x \tan x \, dx = \int (1-\cos^2 x) \tan x \, dx = \int \tan x \, dx - \int \cos^2 x \tan$

$= \int \tan x \, dx - \int \cos^2 x \frac{\sin x}{\cos x} dx = \int \tan x \, dx - \int \cos x \sin x \, dx$

$= \ln|\cos x| + \frac{\cos^2 x}{2} + c.$

2- $\int \cos^3 x \sqrt{\csc x} \, dx = \int \cos^3 x \frac{1}{\sin^{1/2} x} dx = \int \cos^3 x \sin^{-1/2} x \, dx$

$= \int (1-\sin^2 x) \cos x \sin^{-1/2} x \, dx = \int \cos x \sin^{-1/2} x \, dx - \int \sin^{3/2} x \cos x \, dx$

$= 2 \sin^{1/2} x - \frac{2}{5} \sin^{5/2} + C$

3- $\int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = \int \csc^2 x \, dx - \int dx = -\cot x - x + C$

$\int \cot^3 x \, dx = \int \cot^2 x \cot x \, dx = \int (\csc^2 x - 1) \cot x \, dx = \int \csc^2 x \cot x - \int \cot x \, dx$

$= -\frac{\cot^2 x}{2} - \ln|\sin x| + C.$

$\int x^2 \ln x \, dx$, $u = \ln x$, $dv = x^2 dx$ ...

$\int x\sqrt{1+x}\, dx$, $u = x$, $dv = \sqrt{1+x}\, dx$ ...

4) $\int \csc^4 x \, dx = \int \csc^2 x \csc^2 x \, dx = \int (1+\cot^2 x)\csc^2 x \, dx$

$= \int \csc^2 x \, dx + \int \cot^2 x \csc^2 x \, dx = -\cot x - \frac{\cot^3 x}{3} + c.$

5) show that

$$\int \sin^n ax \, dx = -\frac{\sin^{(n-1)} ax \cos ax}{an} + \frac{n-1}{n}\int \sin^{n-2} ax \, dx$$

$$\int \cos^n ax \, dx = \frac{\sin ax \cos^{n-1} ax}{an} + \frac{n-1}{n}\int \cos^{n-2} ax \, dx$$

$$\int \tan^n ax \, dx = \frac{\tan^{n-1} ax}{(n-1)a} - \int \tan^{n-2} ax \, dx.$$

(2) Integration by substitution $\boxed{\text{التكامل بالتعويض :}}$

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Consider the definite integral $\int_{x=a}^{b} f(x)\,dx$

Suppose we wish to make the substitution $x = g(u)$ for some function g. Then there are three things that must be done to change the integral from x variable to u variable

(i) $f(x)$ is replaced by $f(g(u))$

(ii) dx is replaced by $g'(u)\,du$ (since $dx = g'(u)\,du$)

(iii) If c & d are numbers s.t. $a = g(c)$, $b = g(d)$, then a is replaced by c and b replaced by d.

<u>EX!</u> $\int (5x^3+3x-1)^{3/2}(5x^2+1)\,dx$

let $u = 5x^3+3x-1 \Rightarrow du = (15x^2+3)\,dx = 3(5x^2+1)\,dx$

$(5x^2+1)\,dx = \frac{du}{3}$

$\int u^{3/2}\frac{du}{3} = \frac{1}{3}\int u^{3/2} du = \frac{1}{3}\frac{u^{5/2}}{5/2} + c$

$= \frac{2}{15}u^{5/2} + c$

$= \frac{2}{15}(5x^3+3x-1)^{5/2} + c$

Ex: $\displaystyle\int \frac{\sqrt{x}\,dx}{1+\sqrt[4]{x}}$, let $u=\sqrt[4]{x} \Rightarrow x = u^4 \Rightarrow dx = 4u^3\,du$

$\sqrt{x} = u^2$

$$\int \frac{\sqrt{x}\,dx}{1+\sqrt[4]{x}} = \int \frac{u^2\, 4u^3\,du}{1+u} = \int \frac{4u^5\,du}{1+u}$$

$$= 4\int \left(u^4 - u^3 + u^2 - u + 1 - \frac{1}{1+u}\right) du$$

$$= 4\left[\frac{u^5}{5} - \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + C$$

$$= \frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2x^{1/2} + 4x^{1/4} - \ln|x^{1/4}+1| + C$$

$$\begin{array}{r} u^4 - u^3 + u^2 - u + 1 \\ 1+u \;\overline{)\; u^5 \qquad\qquad\qquad} \\ \mp u^5 \mp u^4 \\ \hline -u^4 \\ \pm u^4 \pm u^3 \\ \hline u^3 \\ \mp u^3 \mp u^2 \\ \hline -u^2 \\ \pm u^2 \pm u \\ \hline u \\ \mp u \mp 1 \\ \hline -1 \end{array}$$

Ex: $\displaystyle\int x^3 \sqrt{x^2+a^2}\,dx$

$u = \sqrt{x^2+a^2} \Rightarrow u^2 = x^2 + a^2 \Rightarrow 2u\,du = 2x\,dx$

$$\int x^3\sqrt{x^2+a^2}\,dx = \int x^2 \sqrt{x^2+a^2}\cdot x\,dx$$

$$= \int (u^2-a^2)\,u\cdot u\,du = \int u^4\,du - \int a^2u^2\,du$$

$$= \frac{u^5}{5} - \frac{a^2}{3}u^3 + C$$

$$= \frac{(x^2+a^2)^{5/2}}{5} - \frac{1}{3}a^2(x^2+a^2)^{3/2} + C$$

Ex: $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$; $x = \sin u \Rightarrow dx = \cos u\,du$

If $x = 0 \Rightarrow 0 = \sin u \Rightarrow u = \sin^{-1} 0 = 0$

If $x = 1 \Rightarrow 1 = \sin u \Rightarrow u = \sin^{-1} 1 = \frac{\pi}{2}$

$$\int_{x=0}^{x=1} \sqrt{1-x^2}\,dx = \int_{u=0}^{u=\pi/2} \sqrt{1-\sin^2 u}\,\cos u\,du = \int_0^{\pi/2} \cos^2 u\,du$$

$$= \cdots$$

Ex: $\displaystyle\int_{x=1}^{x=64} \frac{(x+1)\,dx}{\sqrt{x}(1+\sqrt[3]{x})}$ $\qquad u=\sqrt[6]{x} \Rightarrow u^6=x$

$, x=u^6$, then $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$

and $dx = 6u^5\,du$

If $x=1 \Rightarrow u=1$, If $x=64 \Rightarrow x=2^6=u^6 \Rightarrow u=2$

$$\int_{x=1}^{x=64} \frac{(x+1)}{\sqrt{x}(1+\sqrt[3]{x})}dx = \int_{u=1}^{u=2} \frac{6(u^6+1)}{u^3(1+u^2)}u^5\,du = \int_{u=1}^{u=2} \frac{6(u^6+1)u^2\,du}{(1+u^2)}$$

$$=6\int_{u=1}^{u=2} \frac{(u^8+u^2)}{(1+u^2)}du = 6\int_{u=1}^{u=2}(u^6-u^4+u^2)\,du$$

$$= 6\frac{u^7}{7} - \frac{6u^5}{5} + \frac{6}{3}u^3\Big|_1^2 = \frac{6}{7}u^7 + \frac{6}{5}u^5 + 2u^3\Big|_1^2 = \ldots$$

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3) Integrals involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$

((Trigonometric substitutions))

(i) $\sqrt{a^2-x^2}$, set $x=a\sin\theta$, then

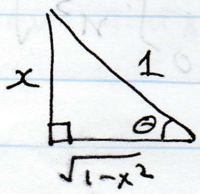
$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\cos\theta$

(ii) $\sqrt{a^2+x^2}$, $x=a\tan\theta$, then

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sec\theta$.

(iii) $\sqrt{x^2-a^2}$, set $x=a\sec\theta$, then

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$



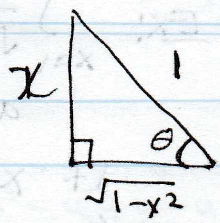
Ex: $\int\sqrt{1-x^2}\,dx$, $x=\sin\theta$, $\Rightarrow dx = \cos\theta\,d\theta$

$$\int\sqrt{1-x^2}\,dx = \int\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta = \int\cos^2\theta\,d\theta$$

$$=\int\frac{1+\cos 2\theta}{2}d\theta = \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + C.$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$x = \sin\theta \Rightarrow \theta = \sin^{-1}x$,



$$\int \sqrt{1-x^2}\,dx = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$$

$$= \frac{\sin^{-1}x}{2} + \frac{\cancel{2}x\sqrt{1-x^2}}{2\,\cancel{4}} + c$$

$\int_{x=0}^{x=1} \sqrt{1-x^2}\,dx$, $x = \sin\theta \Rightarrow 0 = \sin\theta \Rightarrow \sin^{-1}0 = \theta$

$\theta = 0$, if $x = 1$, $1 = \sin\theta \Rightarrow \sin^{-1}1 = \theta \Rightarrow \theta = \frac{\pi}{2}$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Big|_0^{\pi/2} = \frac{1}{2}\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{4} = \frac{\pi}{4}$$

<u>Ex</u>: calculate

$$\int_0^4 x^3\sqrt{16-x^2}\,dx$$

$x = 4\sin\theta$, then $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = 4\cos\theta$

$dx = 4\cos\theta\,d\theta$

limits of integration become

$x=0 \Rightarrow 0 = 4\sin\theta \Rightarrow 0 = \sin\theta \Rightarrow \sin^{-1}0 = \theta$

$\Rightarrow \theta = 0$

$x = 4 \Rightarrow 4 = 4\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \sin^{-1}(1) = \theta$

$\theta = \frac{\pi}{2}$

$$\int_0^4 x^3\sqrt{16-x^2}\,dx = \int_{\theta=0}^{\theta=\pi/2} 4^3\sin^3\theta \cdot 4\cos\theta \cdot 4\cos\theta\,d\theta$$

$$= 4^5\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

$$= 1024\int_0^{\pi/2} (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$$

$$= 1024\left[\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta - \int_0^{\pi/2} \cos^4\theta\sin\theta\,d\theta\right]$$

$$= 1024\left(-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_0^{\pi/2} = 1024\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2048}{15}$$