محاضرات

الاستاذ المساعد

موسى مكي كريدي

المرحلة الأولى

تفاضل وتكامل

II

الفصل الدراسي الثاني

٢٠١٦ ـ ٢٠١٧

التكامل

"Integration"

Antiderivative: عكس المشتقة

يقصد بعكس المشتقة ايجاد الدالة التي علمت مشتقتها.
اذا كانت $F(x)$ داله ما مشتقتها في فترة معينة من x
هي $F'(x)=f(x)$ فاننا نسمي $F(x)$ عكس المشتقة $f(x)$ او
نسميها (اي $F(x)$) التكامل الغير المحدود indefinite Integral
لـ $f(x)$ ويستعمل الرمز $\int f(x)dx$ للدلالة على ان المطلوب
هو التكامل الغير المحدود لـ $f(x)$ ايجاده؛

$$\int f(x)\,dx = F(x) + c$$

Where c is constant of integration (arbitrary constant)

EX: Find an intiderivative of $f(x)=x^2$.

Notice that $F(x)=\frac{1}{3}x^3$ is an intiderivative of $f(x)$. Since

$$F'(x)=\frac{d}{dx}\left(\frac{1}{3}x^3\right)=x^2.$$

but

$$\frac{d}{dx}\left(\frac{1}{3}x^3+1\right)=x^2$$ for any constant

c, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3+c\right)=x^2$$

$H(x)=\frac{1}{3}x^3+c$ is an intiderivative of $f(x)$, for any choice of constant c.

In general: observe that if F is any antiderivative of f and C is any constant, then

$$\frac{d}{dx}[F(x)+C] = F'(x) + 0 = f(x).$$

Theorem: (1) Suppose that F and G are both antiderivative of f on $[a,b]$. Then

$G(x) = F(x) + C$ for some constant C.

Def^n: (1)
Let F be any antiderivative of f. The indefinite integral of $f(x)$ (with respect to x) is defined by

$$\int f(x)\,dx = F(x) + C$$

Where C is an arbitrary constant (constant of integration)

EX: Evaluate $\int 3x^3\,dx = \frac{3}{4}x^4 + C$

$3x^3$ is derivative of $\frac{3}{4}x^4$.

Theorem: (2) (power rule)

For any rational power $r \neq -1$

$$\boxed{\int x^r\,dx = \frac{x^{r+1}}{r+1} + C}$$

EX: $\int x^6\,dx = \frac{x^7}{7} + C$; $\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$

$$\int \frac{dx}{x^3} = \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$$

$$\int \sqrt[3]{x}\,dx = \int 1/x^{1/3}\,dx = \int x^{-1/3}\,dx = \frac{3}{2}x^{2/3} + C$$

Theorem: (3) Suppose that $f(x)$ & $g(x)$ have antiderivatives. Then for any constant a and b

$$\int [a f(x) + b g(x)]\, dx = a\int f(x)\,dx + b\int g(x)\,dx$$

proof:
we have $\frac{d}{dx}\int f(x)\,dx = f(x)$, and $\frac{d}{dx}\int g(x)\,dx = g(x)$, then

$$\frac{d}{dx}\left[a\int f(x)\,dx + b\int g(x)\,dx\right] = a f(x) + b g(x).$$

EX: $\int (3x^2+5)\,dx = \int 3x^2\,dx + \int 5\,dx = x^3 + 5x + C$

Basic Formulas of Integration

1- $\int dx = x + C$ 2- $\int a\,dx = a\int dx$

3- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$, 4. $\int u^n\,du = \frac{u^{n+1}}{n+1},\ n \neq -1$

u is a f^n of x

Or, if $u = f(x) \Rightarrow du = f'(x)\,dx$, then (4) becomes

$$\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C,\ n \neq -1$$

EX: Calculate $\int (x^2+2x-5)^3 (x+1)\,dx$

$u = f(x) = (x^2+2x-5) \Rightarrow du = f'(x) = 2x+2$
$= 2(x+1)\,dx$

$$\int (x^2+2x-5)^3 (x+1)\,dx = \tfrac{1}{2}\int u^3 \cdot du = \frac{1}{2}\frac{u^4}{4} + C$$
$$= \frac{1}{8}(x^2+2x-5)^4 + C$$

(2) $\int (3x^2-9x+1)^{1/2}(2x-3)\,dx$

let

$$u = 3x^2-9x+1 \Rightarrow du = (6x-9)\,dx = 3(2x-3)\,dx$$

$$\int (3x^2-9x+1)^{1/2}(2x-3)\,dx = \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\frac{2}{3}u^{3/2} + C = \frac{2}{9}(3x^2-9x+1)^{3/2} + C$$

(3) find $\int \frac{3x^3+5}{x^2}\,dx = \int \frac{3x^3}{x^2}\,dx + \int \frac{5}{x^2}\,dx$

$$= \int 3x\,dx + \int 5x^{-2}\,dx$$

$$= \frac{3}{2}x^2 + C_1 + 5\frac{x^{-1}}{-1} + C_2$$

$$= \frac{3}{2}x^2 - 5/x + C$$

(4) Solve

$$\frac{dy}{dx} = x\sqrt[3]{y} \quad \text{(Diff. equation)}$$

$$dy = x\sqrt[3]{y}\,dx = x y^{1/3}\,dx$$

$$dy = x y^{1/3}\,dx \Rightarrow \int dy = \int x y^{1/3}\,dx$$

$$\int \frac{dy}{y^{1/3}} = \int x\,dx \Rightarrow \int y^{-1/3}\,dy = \int x\,dx$$

$$\frac{3}{2}y^{2/3} + C_1 = \frac{x^2}{2} + C_2$$

$$3y^{2/3} + 2C_1 = x^2 + 2C_2$$

$$3y^{2/3} - x^2 + C = 0$$

<u>Def^n</u>: For $x>0$, we define the natural logarithm function by $\ln x = \int_1^x \frac{1}{t}\,dt$.

$$\int \sin x\,dx = -\cos x + c \quad ; \quad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c \quad ; \quad \int \csc^2 x\,dx = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c \quad ; \quad \int \csc x \cot x\,dx = -\csc x + c$$

$$\int \frac{dx}{x} = \ln|x| + C,\ x \neq -1 \ ; \int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad ; \quad \int e^{-x} dx = -e^{-x} + C$$

Ex: ①

$$\int (3\cos x + 4x^5)\,dx = \int 3\cos x\,dx + \int 4x^5\,dx$$
$$= 3\int \cos x\,dx + 4\int x^5\,dx$$
$$= 3\sin x + 4\frac{x^6}{6} + C = 3\sin x + \frac{2}{3}x^6 + C$$

②

$$\int (3e^x - 2\sec^2 3x)\,dx = \int 3e^x dx - \int 2\sec^2 3x\,dx$$
$$= 3\int e^x dx - 2\int \sec^2 3x\,dx$$
$$= 3e^x - 2\,\frac{1}{3}\int 3\sec^2 3x\,dx = 3e^x - \frac{2}{3}\tan 3x + C$$

Theorem: if $\int f(x)\,dx = F(x) + C$, then for any constant $a \neq 0$

$$\int f(ax)\,dx = \frac{1}{a}\int a f(ax)\,dx$$
$$= \frac{1}{a}F(ax) + C$$

Ex: $\int \sin 4x\,dx = \frac{1}{4}\int 4\sin 4x\,dx = \frac{-1}{4}\cos 4x + C$

$$\int e^{3x} dx = \frac{1}{3}\int e^{3x}\,3\,dx = \frac{1}{3}e^{3x} + C$$

$$\int 8\sec^2 5x\,dx = \frac{8}{5}\tan 5x + C$$

Corollary: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C, \quad f(x) \neq 0$

Ex: (1) $\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C$

(2) $\int \frac{2x\,dx}{x^2+3} = \ln|x^2+3| + C$

Ex: If a space shuttle's downward acceleration is given by $y''(t) = -32$ ft/s, find the position function $y(t)$. Assume that the shuttle's initial velocity is $y'(0) = -100$ ft/s, and that its initial position is $y(0) = 100{,}000$ feet.

$y'(t) = \int y''(t)\,dt = \int -32\,dt = -32t + C$

$y'(t)$ is the velocity of the shuttle

$v(t) = y'(t) = -32t + C \quad , \quad v(t) = -100$ ft/s

$-100 = v(0) = -32(0) + C \Rightarrow C = -100$

$y(t) = \int y'(t)\,dt = \int (-32t - 100)\,dt$

$= -16t^2 - 100t + C$

$y(t)$ is hight of the shuttle,

$100{,}000 = y(0) = -16(0) - 100(0) + C =$

$C = 100{,}000$

$y(t) = -16t^2 - 100t + 100{,}000$

Ex: $\int 5^{\sin 2x} \ln 5 \cos 2x\,dx = \frac{1}{2}\, 5^{\sin 2x} + C$

$\int x\, 2^{-x^2}\,dx = -\frac{1}{2}\int -2x\, 2^{-x^2}\,dx = -\frac{1}{2\ln 2}\, 2^{-x^2} + C$

Ex: find the function $f(x)$ satisfying the given conditions:

(i) $f'(x) = 4x^2 - 1, \quad f(0) = 2$

(ii) $f''(x) = 12, \quad f'(0) = 2, \quad f(0) = 3$

(iii) $f''(x) = 2x; \quad f'(0) = -3, \quad f(0) = 2$

Sol

Soln (i) $y(x) = \int f'(x)\,dx = \int (4x^2 - 1)\,dx = \frac{4}{3}x^3 - x + C$

since $f(0) = 2$, Hence

$$y(0) = \frac{4}{3}(0) - (0) + C = 2 \Rightarrow C = 2$$

$$\boxed{y(x) = \frac{4}{3}x^3 - x + 2}$$

Ex: find the function satisfying the given conditions

(i) $f''(x) = 3\sin x + 4x^2$, (ii) $f'''(x) = \sin 2x - e^x$.

Soln (i)

$$f'(x) = \int f''(x)\,dx = \int (3\sin x + 4x^2)\,dx$$

$$= -3\cos x + \frac{4}{3}x^3 + c_1$$

$$f(x) = \int f'(x)\,dx = \int [-3\cos x + \frac{4}{3}x^3 + c_1]\,dx$$

$$= -3\sin x + \frac{1}{3}x^4 + c_1 x + c_2$$

$$f''(x) = \int f'''(x)\,dx = \int (\sin 2x - e^x)\,dx = -\frac{1}{2}\cos 2x - e^x + c_1$$

$$f'(x) = \int f''(x)\,dx = -\frac{1}{2}\int \cos 2x - \int e^x\,dx + \int c_1\,dx$$

$$= -\frac{1}{4}\sin 2x - e^x + c_1 x + c_2$$

$$f(x) = \int f'(x)\,dx = -\frac{1}{4}\int \sin 2x\,dx - \int e^x\,dx + c_1\int x + \int c_2\,dx = \frac{1}{8}\cos 2x - e^x + c_1 x^2/2 + c_2 x + c_3$$