Tautology:-

is a formula or assertion that is true in every possible interpreation for example $x = y$ or $x \neq y$

$p: x = y$	$q: x \neq y$	$p \vee q$
T	F	T
F	T	T

Tautology

i.e.; tautology is a statement that is always true

Contradiction:-

is a statement that is always false

p	q	$p \wedge q$
T	F	F
F	T	F

logical equivalence:

the two statements are logically equivalent if they have the same truth table. for example if $P = p$ and $Q = \sim(\sim p)$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

that is, $P \equiv Q$

logical Conclusion:-

let P and Q are phrases, we say that P be a logically requiring from Q or Q implies logically from P and written by $P \Rightarrow Q$ if $P \to Q$ tautology and if Q isn't conclude from P and written by $P \not\Rightarrow Q$

Ex:-

let $P: (p \to q) \wedge (q \to r)$

$Q: p \to r$

Is $P \Rightarrow Q$?

Sol:-

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$P \to Q$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$\therefore P \Rightarrow Q$ because $P \to Q$ is a tautology

Definition:-

let p and Q be aphrases then:

① $Q \rightarrow p$ be the converse of $p \rightarrow Q$

② $\sim Q \rightarrow \sim p$ be the Contrapostive of $p \rightarrow Q$.

Theorem:-

let p and Q be two Statement

① $p \Rightarrow Q$ iff $\sim p \vee Q$ is atautology.

② $p \Rightarrow Q \wedge Q \Rightarrow p$ then $p \equiv Q$.

proof:-

① let $p \Rightarrow Q$ we have to proof that $\sim p \vee Q$ be atautology

Since $p \Rightarrow Q$ then $p \rightarrow Q$ is atautology

then $p \rightarrow Q \equiv \sim p \vee Q$ (i.e.,)

then $\sim p \vee Q$ is atautology

Conversly:-

let $\sim p \vee Q$ be atautology we have to proof that $p \Rightarrow Q$

(i.e.; $p \Rightarrow Q$ is atautology)

Since $\sim p \vee Q$ is atautology

then $p \rightarrow Q \equiv \sim p \vee Q$ and since $\sim p \vee Q$ is atautol-
ogy then $p \rightarrow Q$ is atautology and $p \Rightarrow Q$

Ex:-

Prove that $p \Rightarrow Q$ iff $\sim Q \Rightarrow \sim p$

Sol:-

Since $p \Rightarrow Q$ then $p \to Q$ is a tautology

Since $\sim Q \to \sim p \equiv p \to Q$

Then $\sim Q \to \sim p$ is a tautology that is $\sim Q \Rightarrow \sim p$

Conversly:-

let $\sim Q \Rightarrow \sim p$

Since $\sim Q \Rightarrow \sim p$ is a tautology then $\sim Q \to \sim p$ is a tautology

$\therefore p \to Q \equiv \sim Q \to \sim p$ that is $p \to Q$ is a tautology

thus $p \Rightarrow Q$

Algebra of statements

Suppose that p and Q be statements then:

1- Idempotent property

ⓐ $p \wedge q \equiv p$ ⓑ $p \vee p \equiv p$

2- Commitativity property

ⓐ $p \vee Q \equiv Q \vee p$ ⓑ $p \wedge Q \equiv Q \wedge p$

3- Associativity property

ⓐ $(p \vee Q) \vee R \equiv p \vee (Q \vee R)$

ⓑ $(p \wedge Q) \wedge R \equiv p \wedge (Q \wedge R)$

4- Distributivity property:-

ⓐ $p \vee (Q \wedge R) \equiv (p \vee Q) \wedge (p \vee R)$

ⓑ $p \wedge (Q \vee R) \equiv (p \wedge Q) \vee (p \wedge R)$

5- Identity property:-

ⓐ $p \vee I \equiv I$ ⓑ $p \vee 0 \equiv p$ s.t I is a tautology and 0 is a contradication

6- Complementarity property:-

ⓐ $p \vee \sim p \equiv I$ ⓑ $p \wedge \sim p \equiv 0$ ⓒ $\sim(\sim p) \equiv p$

ⓓ $\sim I \equiv 0$ ⓔ $\sim 0 \equiv I$

s.t I is a tautology statement and 0 is a contradication statement.

7- Demorgen laws:-

ⓐ $\sim(p \wedge Q) \equiv \sim p \vee \sim Q$

ⓑ $\sim(p \vee Q) \equiv \sim p \wedge \sim Q$

Remarks:-

① we can proof any of the above properties using truth table

② $(p \to q) \equiv (\sim p \vee q)$ (an established fact) حقيقة ثابتة

Ex:-

prove without using truth table the following statement

$\sim(p \to q) \equiv p \wedge \sim q$

proof:-

since $p \to q \equiv \sim p \vee q$ then $\sim(p \to q) \equiv \sim(\sim p \vee q)$

$\sim(p \to q) \equiv p \wedge \sim q$

Existential quantifier and universal quantifiers

Existential quantifier:-

Suppose that $p(x)$ is open sentence in X over the set A then the statement there exist $x \in A$ s.t $p(x)$ is true) is called existential quantifier and denoted by $\exists x \in A \ni p(x)$ is true and the symboles $\exists$ is there exist and $\ni$ is (s.t) such that;

Universal property

Suppose that $p(x)$ is an open set in X over A then then statement for all $x \in A$ s.t $p(x)$ is true is called universal quantifiers and denoted by $\forall x \in A \ni p(x)$ is true and the symble $\forall$ is for all.

Ex:-

($\forall x \in \mathbb{R} \ni x^2 \geq 0$) is a universal quantifiers and true

Ex:

There exist $x \in \mathbb{N}$ s.t $x = 7$

: existential quantifier and true

Ex:-

$\exists x \in \mathbb{N} : x + 2 = 7$

existential quantifier and true

Ex:-

$\exists x \in \mathbb{N} : x^2 = 1 \wedge x > 4$

existential quantifier and false

(1) Negation of statement which Containing of universal quantifiers

$$(\forall x \in A : p(x)) \equiv \exists x \in A : \sim p(x)$$

(2) Negation of statement which Containing of existential quantifier

$$(\exists x \in A : p(x)) \equiv \forall x \in A : \sim p(x).$$

Ex:-

What is the negation of the following statement:

$$\forall x \in R : (x^2 - 1 > 0) \vee (x^2 + x > 0 \rightarrow x > 0)$$

$$\exists x \in R : (x^2 - 1 \leq 0) \wedge (x^2 + x \leq 0 \nrightarrow x \leq 0$$

Def:-

Argument: let $S_1, S_2, ---, S_n$ be a statement and S be a statement conclu from the statement $S_1, S_2, --, S_n$ then S be a argument and denoted by $S_1, S_2, ..., S_n \vdash S$

Remark:- The argument $S_1, S_2, ..., S_n \vdash S$ be true iff $S_1 \wedge S_2 \wedge S_3 --- \wedge S_n \rightarrow S$ is a tautology

① $p \vee \sim q,\ p,\ \sim q \vdash p \vee q$

Sol:-

$S_1 = p \vee q$, $S_2 = \sim q$, $S = p \vee q$

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee q$	$S_1 \wedge S_2$	$S_1 \wedge S_2 \rightarrow S$
T	T	F	F	T	T	F	F
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	F
F	F	T	T	T	F	T	T