Equivalence class

Assume R be equivalence class over anonempty set A and let $a \in A$, the set of all elements that related with a by R by equivalence class Contain a and denoted by $[a]$ or A_a

i.e.; $[a] = \{x \in A \,/\, (x,a) \in R\} \subseteq A$

The set of all equivalence class be a quotient set and denoted by A/R

i.e.; $A/R = \{[a] : a \in A\}$

Ex:-

Assume $A = \{1,2,3,4\}$ and $R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1)\}$ and R be an equivalence relation over A find A/R

Sol:-

$[a] = \{x \in A \,/\, (x,a) \in R\}$

$A/R = \{[a] : a \in A\}$

$[1] = \{x \in A \,/\, (x,1) \in R\} = \{1,3\}$

$[2] = \{x \in A \,/\, (x,2) \in R\} = \{2\}$

$[3] = \{x \in A \,/\, (x,3) \in R\} = \{3,1\} = [1]$

$[4] = \{x \in A \,/\, (x,4) \in R\} = \{4\}$

$A/R = \{[1],[2],[4]\}$

since $[1] = [3]$

Remark:-

we note that $[1] \cup [2] \cup [4] = A$

$\{1,3\} \cup \{2\} \cup \{4\} = \{1,2,3,4\} = A$

Ex:-

Assume Z be a set of integer numbers and R be a set over Z

defined by $R = \{(x,y) \in Z \times Z / x-y \text{ divisible by } 3\}$

Find Z/R

Sol:-

$[0] = \{x \in Z / (x,0) \in R\}$ s.t $R = \{(x,y) \in Z \times Z / x-0 \text{ divisible by } 3\}$

$\{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$[1] = \{x \in Z / (x,1) \in R\}$ s.t $R = \{(x,y) \in Z \times Z / x-1 \text{ divisible by } 3\}$

$\{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$[2] = \{x \in Z / (x,2) \in R\}$ s.t $R = \{(x,y) \in Z \times Z / x-2 \text{ divisible by } 3\}$

$\{\ldots, -7, -4, -1, 2, 5, 8, 11, 14, \ldots\}$

$[3] = \{x \in Z / (x,3) \in R\}$ s.t $R = \{(x,y) \in Z \times Z / x-3 \text{ divisible by } 3\}$

$\{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

$[4] = \{x \in Z / (x,4) \in R\}$ s.t $R = \{(x,y) \in Z \times Z / x-4 \text{ divisible by } 3\}$

$\{\ldots, -5, -2, 1, 4, 7, \ldots\}$

$[3] = [0]$, $[4] = [1]$, $[5] = [2]$

$[6] = [5] = [0]$, $[7] = [1]$ ---

$Z/R = \{[0], [1], [2]\}$

Theorem:-

Assume R be a relation over A then; R is a symmetric relation iff $R = R^{-1}$

Proof:- $(\Longrightarrow)$

Assume R be asymmetric

To proof that $R = R^{-1}$, we must proof that $R \subseteq R^{-1}$

and $R^{-1} \subseteq R$

Now, we must proof that $R \subseteq R^{-1}$

let $(x,y) \in R$

Since R is asymmetric relation then $(y,x) \in R$

then $(x,y) \in R^{-1}$

so $R \subseteq R^{-1}$ --- (1)

Now, we must proof that $R^{-1} \subseteq R$

let $(x,y) \in R^{-1}$

then $(y,x) \in R$

since R is asymmetric relation

So $(x,y) \in R$

that is $R^{-1} \subseteq R$ --- (2)

From (1) and (2), we get $R = R^{-1}$

$(\Longleftarrow)$

Assume $R = R^{-1}$ and we must proof that R is a symmetric relation

let $x, y \in A$ s.t $(x, y) \in R$

Since $R = R^{-1}$ then $(x, y) \in R^{-1}$ then $(y, x) \in R$

So that R be a symmetric relation

properties of equivalence classes

خواص صفوف التكافؤ

Theorem :-

Assume R be an equivalence relation over A and let $a, b \in A$ then :-

① $a \in [a]$

② If $b \in [a]$ then $[a] = [b]$

③ $[a] = [b]$ iff $(a, b) \in R$

proof :-

① From the defination of equivalence relation

$[a] = \{x \in A / (x, a) \in R\}$

since R be an equivalence relation

then R be a reflexive relation

So that $a \in A$ then $(a,a) \in R$, that is $a \in [a]$

2) To proof that $[a] = [b]$

we must proof that $[a] \subseteq [b]$ and $[b] \subseteq [a]$

let $x \in [a] \longrightarrow (x,a) \in R$

since $b \in [a] \longrightarrow (b,a) \in R$

$(x,a) \in R \wedge (b,a) \in R$

Since R is an equivalence relation

then R is asymmetric relation

then $(x,a) \in R \wedge (a,b) \in R$

since R be atranstive relation

then $(x,b) \in R \longrightarrow x \in [b]$

then $[a] \subseteq [b]$ -- (1)

Now, To proof that $[b] \subseteq [a]$

let $x \in [b] \longrightarrow (x,b) \in R$

Since $b \in [a] \longrightarrow (b,a) \in R$

we get $(x,b) \in R \wedge (b,a) \in R$

Since R is an equivalence relation

R is a transtive relation

then $(x,a) \in R$

then $x \in [a]$

so that $[b] = [a]$ ... (2)

From (1) and (2), we get $[a]=[b]$

3) Assume $[a] = [b]$

To proof that $(a,b) \in R$

Since $a \in [a]$ and $[a] = [b]$ then $a \in [b]$

$(a,b) \in R$

Conversely:-

Assume $(a,b) \in R$

Now, To proof that $[a] = [b]$

then $(a,b) \in R$ then $a \in [b]$

from (2), we get $[a] = [b]$.

The partition التجزئة

Assume $\{A_i\}_{i\in I}$ be anonempty family of a set A iff $\{A_i\}_{i\in I}$ be apartition of the set A if it is Satisfies the following conditions:-

① $A = \bigcup_{i\in I} A_i$ ② $A_i \cap A_j = \phi \quad \forall i \neq j$

Ex:-

let $A=\{1,2,3\}$ Determine wether the following relations be a family of sets which it is a partitation of the set A or not?

① $R=\{\{1,2\},\{2,3\}\}$

② $S=\{\{1\},\{2\},\{3\}\}$

Sol:-

① Assume $A_1=\{1,2\}$, $A_2=\{2,3\}$

$A_1 \cup A_2 = A$

$A_1 \cap A_2 = \{2\} \neq \phi$

$\therefore$ R isn't a partition of A

2) Assume $A_1=\{1\}$, $A_2=\{2\}$, $A_3=\{3\}$

then $\bigcup_{i\in I} A_i = A_1 \cup A_2 \cup A_3 = A$

$A_1 \cap A_2 = \phi$ and $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$

Then S is apartition of A.

"partial ordered relation"

علاقة الترتيب الجزئي

Assume R be an equivalence relation over A, we said that R be a partial ordered relation if itis Satisfies the following Conditions:-

1) R is areflexive

2) R is an anti-Symmetric relation

3) R is an transtive relation

Ex:-

Assume be aset of integer numbers. and let R_i s.t $i=1,2$

$R_1=\{(x,y)\in Z\times Z / x\leq y\}$

$R_2=\{(x,y)\in Z\times Z / x-y \text{ divisible } 2\}$. IS R_1 and R_2 be partial orderset.

Sol:-

* $R_1=\{(x,y)\in Z\times Z / x\leq y\}$

1) Assume $x\in Z$, since $x\leq x$ then $(x,x)\in R$

then R_1 areflexive relation

2) Assume $x,y\in Z$ s.t $(x,y)\in R_1 \wedge (y,x)\in R_1$

Since $(x,y)\in R_1 \rightarrow x\leq y$ --- (1)

Since $(y,x)\in R_1 \rightarrow y\leq x$ --- (2)

From (1) and (2), weget $x=y$

3) Assume $x, y, z \in \mathbb{Z}$ s.t $(x,y) \in R_1 \wedge (y,z) \in R_1$

Since $(x,y) \in R_1 \longrightarrow x \leq y$ --- ①

Since $(y,z) \in R_1 \longrightarrow y \leq z$ --- ②

From (1) and (2), we have $x \leq z$ then $(x,z) \in R_1$

then R_1 is an transtive relation

that is, R is a reflexive, anti-symmetric, and transtive

thus, R_1 is a partial ordered set

* $R_2 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \,/\, x-y \text{ divisible by } 2\}$

R_2 isn't anti- symmetric relation

since $(8,4) \in R_2$; $4-8=-4$ divisible by 2

since $(4,8) \in R_2$: $8-4=4$ divisible by 2

but $8 \neq 4$

that is, R_2 isn't anti- symmetric relation

thus, R_2 isn't partial ordered set

3) Assume $x, y, z \in \mathbb{Z}$ s.t $(x,y) \in R_1 \wedge (y,z) \in R_1$

Since $(x,y) \in R_1 \longrightarrow x \leq y$ --- (1)

Since $(y,z) \in R_1 \longrightarrow y \leq z$ --- (2)

From (1) and (2), we have $x \leq z$ then $(x,z) \in R_1$

then R_1 is an transtive relation

that is, R is a reflexive, anti-Symmetric, and transtive

thus, R_1 is apartial ordered set

* $R_2 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} / x-y$ divisible by $2\}$

R_2 isn't anti- symmetric relation

since $(8,4) \in R_2$; $8-4 = 4$ divisible by 2

since $(4,8) \in R_2$; $4-8 = -4$ divisible by 2

but $8 \neq 4$

that is, R_2 isn't anti- symmetric relation

thus, R_2 isn't partial ordered set