Identity relation العلاقة الذاتية

If $A \neq \phi$ and R be a relation defined over A, we said that R is an identity relation if $\forall (x,y) \in R \longrightarrow x = y$

Ex:-

If $A = \{a,b,c\}$ then $I_A = \{(a,a),(b,b),(c,c)\}$ be an identity relation

The domain and the range of relation
منطلق ومدى العلاقة

let R be a relation from A to B then;

① the set of all first element of pairs in R called by domain R and denoted by dom(R)

i.e.; $dom(R) = \{x / \exists y \in B \ni (x,y) \in R\} \subseteq A$

② The set of all second element of pairs in R called by rang R and denoted by ran(R)

i.e., $Rang(R) = \{y / \exists x \in A \ni (x,y) \in R\} \subseteq B$

Theorem:-

let R be a relation over A then $(R^{-1})^{-1} = R$

proof:- $(\Rightarrow)$

To proof that $(R^{-1})^{-1} \subseteq R$

let $(x,y) \in (R^{-1})^{-1}$

$(y,x) \in R^{-1}$

$(x,y) \in R$

$\therefore (R^{-1})^{-1} \subseteq R$

$(\Leftarrow)$

To proof that $R \subseteq (R^{-1})^{-1}$

Let $(x,y) \in R$

$(y,x) \in R^{-1}$

$(x,y) \in (R^{-1})^{-1}$

$\therefore R \subseteq (R^{-1})^{-1}$

Theorem:-

let R be a relation from A to B then $\text{dom } R = \text{ran } R^{-1}$

proof:- To proof that $\text{dom } R \subseteq \text{ran } R^{-1}$

let $x \in \text{dom}(R)$

$\exists y \in B \ni (x, y) \in R$

$\Rightarrow (y, x) \in R^{-1}$

$\Rightarrow x \in \text{ran } R^{-1}$

$\therefore \text{dom } R \subseteq \text{ran } R^{-1}$ ... ①

To proof that $\text{ran } R^{-1} \subseteq \text{dom } R$

let $x \in \text{ran } R^{-1}$

$\exists y \in B \ni (y, x) \in R^{-1}$

Then $(x, y) \in R$

$x \in \text{dom } R$

$\therefore \text{ran } R^{-1} \subseteq \text{dom } R$ --- ②

from ① and ② we have $\text{dom } R = \text{ran } R^{-1}$

Composition of relation

If R be a relation from A to B and S be a relation B to C

then $S \circ R$ be a relation from A to C defined by:

$S \circ R = \{(x, y) \in A \times C \,/\, \exists z \in B \ni (x, z) \in R \wedge (z, y) \in S\}$

Ex:-

let $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$ and let

$R = \{(1,3), (2,4)\}$ and $S = \{(3,5), (3,6), (4,5)\}$ find $S \circ R$?

Sol:-

Since $(1,3) \in R \wedge (3,5) \in S$ then SoR

$(1,3) \in R \wedge (3,6) \in S$ then $(1,6) \in SoR$

Since $(2,4) \in R \wedge (4,5) \in S$ then $(2,5) \in SoR$

That is, $SoR = \{(1,5), (1,8), (2,5)\}$

Theorem:-

let R be arelation over A then $I_A o R = R o I_A = R$

proof:- $(\Rightarrow)$

To proof that $I_A o R \subseteq R$

let $(x,y) \in I_A o R$

$\exists z \in A \ni (x,z) \in R \wedge (z,y) \in I_A$

Since $(z,y) \in I_A$ then $z = y$

Since $(x,z) \in I_A$ and $(y,y) \in I_A$ then $(x,y) \in R$

that is, $I_A o R \subseteq R$. --- ①

$(\Leftarrow)$

To proof that $R \subseteq I_A o R$

let $(x,y) \in R$

$(x,y) \in R \wedge (y,y) \in I_A$

$(x,y) \in I_A o R$

$R \subseteq I_A o R$ ---- ②

from ① and ② weget $I_A o R = R$.

Theorem:

Assume T, S, and R be a relations over A then:

① $(T \circ S) \circ R = T \circ (S \circ R)$

② $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ ③ $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$

proof:-

① $(T \circ S) \circ R = T \circ (S \circ R)$

$(\Rightarrow)$ To proof that $(T \circ S) \circ R \subseteq T \circ (S \circ R)$

let $(x, y) \in (T \circ S) \circ R$

$\exists z \in A \ni (x, z) \in R \wedge (z, y) \in T \circ S$

$\exists w \in A (x, z) \in R \wedge ((z, w) \in S \wedge (w, y) \in T)$

$(x, z) \in R \wedge (z, w) \in S \wedge (w, y) \in T$

$(w, y) \in T \wedge [(x, z) \in R \wedge (z, w) \in S$

$(w, y) \in T \wedge (x, w) \in S \circ R$

$(x, w) \in S \circ R \wedge (w, y) \in T$

$(x, y) \in T \circ (S \circ R)$, $\Rightarrow (T \circ S) \circ R \subseteq T \circ (S \circ R)$ --- ①

$(\Leftarrow)$ To proof that $T \circ (S \circ R) \subseteq (T \circ S) \circ R$

let $(x, y) \in T \circ (S \circ R)$

$\exists z \in A \ni (x, z) \in S \circ R \wedge (z, y) \in T$

$\exists w \in A \ni (x, w) \in R \wedge (w, z) \in S \wedge (z, y) \in T$

$(z, y) \in T \wedge (w, z) \in S \wedge (x, w) \in R$

$((w, z) \in S \wedge (z, y) \in T) \wedge (x, w) \in R$

$(w,y) \in T \circ S \wedge (x,w) \in R$

$(x,w) \in R \wedge (w,y) \in T \circ S$

$(x,y) \in (T \circ S) \circ R$ --- (2)

$(T \circ S) \circ R = T \circ (S \circ R)$

(2) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

proof:-

To proof that $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$

let $(x,y) \in (S \circ R)^{-1}$ then $(y,x) \in S \circ R$

$\exists z \in A \ni (y,z) \in R \wedge (z,x) \in S$

$\exists z \in A \ni (z,y) \in R^{-1} \wedge (x,z) \in S^{-1}$

$\exists z \in A \ni (x,z) \in S^{-1} \wedge (z,y) \in R^{-1}$

thatis, $(x,y) \in R^{-1} \circ S^{-1}$

$(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$ --- (1)

To proof that $(R^{-1} \circ S^{-1}) \subseteq (S \circ R)^{-1}$

let $(x,y) \in R^{-1} \circ S^{-1}$

$\exists z \in A \ni (x,z) \in S^{-1} \wedge (z,y) \in R^{-1}$

$\exists z \in A \ni (z,x) \in S \wedge (y,z) \in R$

$\exists z \in A \ni (y,z) \in R \wedge (z,x) \in S$

thatis, $(y,x) \in S \circ R$

$(x,y) \in (S \circ R)^{-1}$ --- (2)

From (1) and (2), we have $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

③ To proof $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$

proof:-

let $(x,y) \in (S \cap T) \circ R$

$\exists z \in A \ni (x,z) \in R \wedge (z,y) \in S \cap T$

$(x,z) \in R \wedge (z,y) \in S \wedge (z,y) \in T$

$(x,z) \in R \wedge (z,y) \in S) \wedge (z,y) \in T$

Since $(x,z) \in R \wedge (z,y) \in S$ then $(x,y) \in S \circ R$ -- ①

Since $(x,z) \in R \wedge (z,y) \in T$

then $(x,y) \in T \circ R$ -- ②

thatis, $(x,y) \in (S \circ R) \wedge (x,y) \in T \circ R$

then $(x,y) \in (S \circ R) \cap (T \circ R)$

Hence $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.